NOTES ON FINITE GROUP REPRESENTATIONS

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In Fall 2020, I taught an undergraduate course on abstract algebra. I chose to spend two weeks on the theory of complex representations of finite groups. I covered the basic concepts, leading to the classification of representations by characters. I also briefly addressed a few more advanced topics, notably induced representations and Frobenius divisibility. I'm making the lectures and these associated notes for this material publicly available.

The material here is standard, and is mainly based on Steinberg, *Representation theory of finite groups*, Ch 2-4, whose notation I will mostly follow. I also used Serre, *Linear representations of finite groups*, Ch 1-3.[1]

1. Group representations

Given a vector space V over a field F, we write $GL(V)$ for the group of bijective linear maps $T\colon V \to V$.

When $V = F^n$ we can write $GL_n(F) = GL(F^n)$, and identify the group with the group of invertible $n \times n$ matrices.

A **representation** of a group G is a homomorphism of groups $\phi\colon G \to GL(V)$ for some choice of vector space V. I'll usually write $\phi_g \in GL(V)$ for the value of ϕ on $g \in G$. **representation**

When $V = F^n$, so we have a homomorphism $\phi\colon G \to GL_n(F)$, we call it a **matrix representation**. **matrix representation**

The choice of field F matters. For now, we will look exclusively at the case of $F = \mathbb{C}$, i.e., representations in complex vector spaces.

Remark. Since $\mathbb{R} \subseteq \mathbb{C}$ is a subfield, $GL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{C})$. So any real matrix representation of G is also a complex matrix representation of G.

The **dimension** (or **degree**) of a representation $\phi\colon G \to GL(V)$ is the dimension of the vector space V. We are going to look only at *finite dimensional* representations. (Note: our textbooks prefer the term "degree", but I will usually call it "dimension".) **dimension** **degree**

2. Examples of group representations

Example (Trivial representation). $\phi\colon G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ given by $\phi_g = 1$ for all $g \in G$.

Warning. the set $V = \{0\}$ is a vector space (its 0 dimensional). I'm going to follow convention and sometimes write "0" for this vector space. Then $GL(\{0\})$ is the trivial group. Thus there is always a representation $G \to GL(\{0\})$, which is even more trivial that the trivial representation. If I need to mention this I'll call it the "0-representation".

We write $\mathbb{Z}_n$ for the cyclic group of order n, whose elements are congruence classes $[k] := \{\, x \in \mathbb{Z} \mid x \equiv k \pmod{n} \,\}$, and whose group law is addition.

Example. $\phi\colon \mathbb{Z}_4 \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ by $\phi_{[k]} = i^k$.

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[1] More precisely, I'm following Steinberg, except that I'm avoiding all references to "unitary representations". Where this notion appears in proofs, I'm instead using arguments based on Serre's elegant proofs. (There's nothing wrong with knowing about unitary representations, but it's overkill given that I don't get very far into the material.)

Example. $\phi\colon \mathbb{Z}_4 \to GL_2(\mathbb{C})$ by $\phi_{[k]} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k$.

Example. For any $n \geq 1$, the map $\phi\colon \mathbb{Z}_n \to \mathbb{C}^\times$ by $\phi_{[k]} := e^{2\pi k/n}$.

Example (Standard representation of S_n). $\rho\colon S_n \to GL_n(\mathbb{C})$ defined so that ρ_g is the permutation matrix of g:

$$\rho_g = [e_{g(1)} \; \cdots \; e_{g(n)}],$$

whose k-th column is the standard basis vector $e_{g(k)}$. For instance, $\rho\colon S_3 \to GL_3(\mathbb{C})$ with

$$\rho_{(1\,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho_{(1\,2\,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

etc.

Given any homomorphism $\psi\colon H \to G$ of groups, and a representation $\phi\colon G \to GL(V)$ of G, we get a representation $\phi \circ \psi$ of H:

$$H \xrightarrow{\psi} G \xrightarrow{\phi} GL(V).$$

When $\psi\colon H \rightarrowtail G$ is the inclusion of a subgroup, we write $\phi|_H := \phi \circ \psi$ and call it the **restriction** of ϕ to H. **restriction**

Example. Let $\rho\colon S_3 \to GL_3(\mathbb{C})$ be the standard representation of S_3, and let $\pi\colon S_4 \to S_3$ be a surjective homomorphism (whose existence we have shown earlier: S_3 is isomorphic to the quotient S_4/V where $V \leq S_4$ is the subgroup generated by elements of the form $(a\,b)(c\,d) \in S_4$). This gives a 3-dimensional representation $\phi \circ \pi$ of S_4.

Example. Let $V =$ the set of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. This is an *infinite dimensional* vector space. Define $\phi\colon \mathbb{Z}_2 \to GL(V)$ by

$$\phi_{[k]}(f) := g, \qquad g(x) := g((-1)^k x).$$

This is an example of an infinite dimensional representation of $\mathbb{Z}_2$.

Remark. A representation of G consists of a choice two pieces of data (V, ϕ): a vector space V and a homomorphism $\phi\colon G \to GL(V)$. As a shorthand, I'll usually refer to the represetation by "ϕ", but some people prefer to refer to it by "V".

3. Equivalences of representations

Fix the group G, and let $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(V)$ be representations. An **equivalence** is a linear isomorphism $T\colon V \to W$ of vector spaces such that **equivalence of representations**

$$\psi_g = T \circ \phi_g \circ T^{-1} \qquad \text{for all } g \in G.$$

We write $\phi \sim \psi$ and say the representations are **equivalent** if there exists an equivalence. **equivalent** (Exercise: "equivalence" is an equivalence relation on the collection of representations of G: if T is an equivalence $\phi \sim \psi$, and S is an equivalence $\phi \sim \rho$, then $S \circ T$ is an equivalence $\phi \sim \rho$.)

Example. Define $\phi, \psi\colon \mathbb{Z}_n \to GL_2(\mathbb{C})$ by

$$\phi_{[k]} := \begin{bmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{bmatrix}, \qquad \psi_{[k]} := \begin{bmatrix} e^{2\pi k/n} & 0 \\ 0 & e^{-2\pi k/n} \end{bmatrix}.$$

Then left multiplication by $A := \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$ gives an equivalence: verify that $\psi_g = A\phi_g A^{-1}$.

Let $\phi\colon G \to GL(V)$ be a representation of dimension n. Choose a basis $B = \{v_1, \dots, v_n\}$ of V. We get a vector space isomorphism $T\colon V \to \mathbb{C}^n$ by sending $v \in V$ to its set of coordinates wrt B:

$$T(x_1 v_1 + \cdots + x_n v_n) = (x_1, \dots, x_n), \qquad x_1, \dots, x_n \in F.$$

Let $\psi_g := T\phi_g T^{-1}$. That is, $\psi_g = [\phi_g]_B$, the matrix representing ϕ_g in the basis B. Then $\psi\colon G \to GL_n(\mathbb{C})$ is a homomorphism, and T gives an equivalence of representations $\phi \sim \psi$.

Thus, *every representation is equivalent to a matrix representation.*

Our basic goal is to classify representations of G up to equivalence.

4. Invariant subspaces

Given a representation $\phi\colon G \to GL(V)$, a vector subspace $W \leq V$ is **G-invariant** (or just **invariant**) if $\phi_g(W) = W$ for all $g \in G$, where $\phi_g(W) \subseteq V$ is the image of W under the function ϕ_g. **G-invariant subspace** **invariant subspace**

Given a such a G-invariant subspace, we can restrict ϕ to a representation

$$\phi|_W\colon G \to GL(W),$$

by $(\phi|_W)_g(w) := \phi(w)$. We call $\phi|_W$ a **subrepresentation** of ϕ. **subrepresentation**

Example. Consider the standard representation $\rho\colon S_n \to GL_n(\mathbb{C})$. Let $W = \mathbb{C}v$ where $v = e_1 + \cdots + e_n$. Then W is an invariant subspace, since $\rho_g(v) = e_{g(1)} + \cdots + e_{g(n)} = e_1 + \cdots + e_n$.

The restricted representation $\rho|_W$ is equivalent to the trivial representation.

Example. Consider the standard representation $\rho\colon S_3 \to GL_3(\mathbb{C})$. Let $U = \mathbb{C}x + \mathbb{C}y$, where $x = e_1 - e_2$ and $y = e_2 - e_3$. Then U is an invariant subspace. To see this, it suffices to check that $\rho_g(U) \subseteq U$ for $g \in \{(1\ 2),\ (1\ 2\ 3)\}$. We compute

$$\rho_{(1\ 2)}(x) = -x, \quad \rho_{(1\ 2)}(y) = x + y, \qquad\qquad \rho_{(1\ 2\ 3)}(x) = y, \quad \rho_{(1\ 2\ 3)}(y) = -x - y.$$

The restricted representation $\rho|_U$ is equivalent to a matrix representation $\phi\colon G \to GL_2(\mathbb{C})$ with

$$\phi_{(1\ 2)} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \phi_{(1\ 2\ 3)} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Note: it turns out these are the *only* invariant subspaces of $\rho\colon S_3 \to GL_3(\mathbb{C})$, other than $0 = \{0\}$ and $\mathbb{C}^3$.

5. Irreducible representations

We say that a representation $\phi\colon G \to GL(V)$ is **irreducible** if (i) $V \neq 0$, and (ii) the only G-invariant subspaces are 0 and V. **irreducible representation**

Example. Any 1-dimensional representation is irreducible.

Example. The standard representation $\rho\colon S_3 \to GL_3(\mathbb{C})$ is not irreducible. But the two subrepresentations $\rho|_W$ and $\rho|_U$ that we found turn out to be irreducible.

Example. Here's a proof that the representation $\phi\colon G \to GL_2(\mathbb{C})$ defined earlier is irreducible. Remember that $\phi_{(1\ 2)} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\phi_{(1\ 2\ 3)} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. If this is not irreducible, then there is a 1-dimensional subspace $W \leq \mathbb{C}^2$ which is invariant. Write $W = \mathbb{C}v$ for some $v \in \mathbb{C}^2$. Then $v \neq 0$, and $\phi_g(v) \in \mathbb{C}v$ for all $g \in G$, i.e., v must be an eigenvector for *every* $g \in G$. But we can check explicitly that

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are (up to scalar) the only eigenvectors of $\phi_{(1\ 2)}$, and that neither is an eigenvector of $\phi_{(1\ 2\ 3)}$.

6. Direct sum of representations

Given vector spaces $V_1, \dots, V_n$, their **external direct sum** (or simply **direct sum**) is a vector space $V = V_1 \oplus \cdots \oplus V_n$, whose underlying set is the direct product $V_1 \times \cdots \times V_n$. (You won't confuse anyone if you call it the direct product, but it is usually called "direct sum".)

external direct sum
direct sum

Given representations $\phi^{(k)}: G \to GL(V_k)$, $k = 1, \dots, n$, their **external direct sum** is the representation $\phi: G \to GL(V)$, where

external direct sum

$$V := V_1 \oplus \cdots \oplus V_n, \qquad \phi: G \to GL(V), \qquad \phi_g(x_1, \dots, x_n) := (\phi_g^{(1)}(x_1), \dots, \phi_g^{(n)}(x_n)).$$

Conventionally, we write $\phi = \phi^{(1)} \oplus \cdots \oplus \phi^{(n)}$ for this homomorphism.

Example (Direct sums of matrix representations). If $V_1 = \mathbb{C}^{n_1}$ and $V_2 = \mathbb{C}^{n_2}$, then we can identify $V_1 \oplus V_2$ with $\mathbb{C}^{n_1+n_2}$. Thus, given $\phi^{(j)}: G \to GL_{n_j}(\mathbb{C})$ for $j = 1, 2$, the direct sum representation has block matrix form

$$(\phi^{(1)} \oplus \phi^{(2)})_g = \begin{bmatrix} \phi_g^{(1)} & 0 \\ 0 & \phi_g^{(2)} \end{bmatrix}.$$

Say that V is an **internal direct sum** of subspaces $V_1, \dots, V_n \leq V$ if the map

internal direct sum

$$V_1 \oplus \cdots \oplus V_n \to V, \qquad (x_1, \dots, x_n) \mapsto x_1 + \cdots + x_n$$

is an isomorphism.

Note: if $n = 2$, then V is an internal direct sum of $V_1, V_2 \leq V$ iff $V_1 + V_2 = V$ and $V_1 \cap V_2 = 0$. When $n > 2$ there is also a criterion like this, but it's more complicated to state.

If $\phi: G \to GL(V)$ is a representation, and $V_1, \dots, V_n \leq V$ are G-invariant subspaces, and if V is an internal direct sum of these subspaces, then we say we say that ϕ is an **internal direct sum** of the subrepresentations $\phi|_{V_k}$. In this case, there is an equivalence of representations $(\phi|_{V_1}) \oplus \cdots \oplus (\phi|_{V_n}) \sim \phi$ between ϕ and the external direct sum built from the subrepresentations.

internal direct sum

We say that $\phi: G \to GL(V)$ is **decomposable** if it is an internal direct sum of two non-0 invariant subspaces V_1 and V_2. (I.e., both V_1 and V_2 have positive dimension.)

decomposable

We say that $\phi: G \to GL(V)$ is **completely reducible** if it is equivalent to direct sum of a finite sequence of irreducible subrepresentations.

completely reducible

Proposition. *If $\phi: G \to GL(V)$ and $\psi: G \to GL(W)$ are equivalent representations, then ϕ is irreducible/decomposable/completely reducible iff ψ is.*

Proof. Here is the proof for irreducibility. Let $T: V \to W$ be an equivalence. I just need to show that ψ irreducible implies ϕ irreducible. If ϕ is not irreducible, then there exists an invariant subspace $V' \leq V$ such that $V' \neq 0$ and $V' \neq V$. Let $W' = T(V')$. Then W' is an invariant subspace of W such that $W' \neq 0$ and $W' \neq W$. □

7. Morphisms of representations

Let $\phi: G \to GL(V)$ and $\psi: G \to GL(W)$ be two representations of G. A **morphism** of of representations from ϕ to ψ is a linear map $T: V \to W$ such that

morphism of representations

$$\psi_g \circ T = T \circ \phi_g \qquad \text{for all } g \in G.$$

Such a T is also sometimes called an **intertwining operator**.

intertwining operator

Note that if T is a bijection, then the above identity can be rewritten as

$$\psi_g = T \circ \phi_g \circ T^{-1} \qquad \text{for all } g \in G,$$

so a morphism which is a bijection is exactly what we called an equivalence.

Given vector spaces V, W, I'll write $\mathrm{Hom}(V, W)$ for the set of linear maps $V \to W$. Note that $\mathrm{Hom}(V, W)$ is also a vector space: you can add linear maps, and multiply a linear map by a scalar.

Given representations $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(W)$, write

$$\mathrm{Hom}_G(\phi, \psi) := \{\, T \in \mathrm{Hom}(V, W) \mid \psi_g T = T\phi_g \ \ \forall\, g \in G \,\}$$

for the set of morphisms of representations. Note that this is a vector subspace of $\mathrm{Hom}(V, W)$: if $T_1, T_2 \in \mathrm{Hom}_G(\phi, \psi)$ and $c_1, c_2 \in \mathbb{C}$, then

$$\psi_g(c_1T_1 + c_2T_2) = c_1\psi_gT_1 + c_2\psi_gT_2 = c_1T_1\phi_g + c_2T_2\phi_g = (c_1T_1 + c_2T_2)\phi_g.$$

Note that if V and W are finite dimensional vector spaces, then so is $\mathrm{Hom}(V, W)$, and thus so is $\mathrm{Hom}_G(\phi, \psi)$.

Proposition. *Let $T\colon V \to W$ be a morphism of representations $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(W)$. Then the subspaces $\ker(T) \le V$ and $T(V) \le W$ are invariant subspaces.*

Proof. This is straightforward: If $v \in \mathrm{Ker}(T)$, then $T(\phi_g(v)) = \psi_g(T(v)) = \phi_g(0) = 0$, so $\phi_g(v) \in \mathrm{Ker}(T)$. If $w = T(v) \in T(V)$, then $\psi_g(W) = \psi_g(T(v)) = T(\psi_g(v)) \in T(V)$. □

Exercise. It turns out that if $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(W)$ are representations, then $\mathrm{Hom}(V, W)$ is also a representation! Define

$$\gamma\colon G \to GL(\mathrm{Hom}(V, W))$$

by

$$\gamma_g(T) := \psi_g T \phi_{g^{-1}}.$$

Show that this defines a representation. Then show that $\gamma_g(T) = T$ for all $g \in T$ iff $T \in \mathrm{Hom}_G(\phi, \psi)$.

8. The averaging trick for morphisms

Given representations $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(W)$, and a linear map $T \in \mathrm{Hom}(V, W)$, there is a way to produce from T a morphism $T' \in \mathrm{Hom}_G(\phi, \psi)$ by "averaging T over group elements".

Proposition (Averaging trick)**.** *Let $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(W)$ be representations of a finite group G. Given a linear map $T \in \mathrm{Hom}(V, W)$, define a function $T'\colon V \to W$ by*

$$T' := \frac{1}{|G|}\sum_{g\in G} \psi_g T \phi_g^{-1}.$$

Then $T' \in \mathrm{Hom}_G(\phi, \psi)$.

Proof. First, note that $T'\colon V \to W$ is certainly a linear map, since ψ_g, T, and ϕ_g^{-1} are linear. To show that it is a morphism of representations, we show $\psi_a T' \phi_a^{-1} = T'$ for all $a \in G$.

$$\begin{aligned}
\psi_a T' \phi_{a^{-1}} &= \frac{1}{|G|}\sum_{g\in G} \psi_a\psi_g T \phi_g^{-1}\phi_a^{-1} \\
&= \frac{1}{|G|}\sum_{g\in G} \psi_{ag} T \phi_{ag}^{-1} \\
&= \frac{1}{|G|}\sum_{h\in G} \psi_h T \phi_h^{-1} = T' \qquad\qquad \text{where } h = ag.
\end{aligned}$$

□

Exercise. Show that if $T \in \mathrm{Hom}_G(\phi, \psi)$, then $T' = T$.

Remark. This is the first fact which relies on the fact that G is finite, and on the fact that the field is $\mathbb{C}$ (or at least, has characteristic 0). Clearly, we need G finite so that we can divide by $|G|$ in the averaging formula.

We also need the field F to have "characteristic 0", i.e., we need $1 + \cdots + 1 \neq 0$ in F, so that we can divide by $|G|$. We cannot use the averaging trick for representations over the finite field $F = \mathbb{Z}_p$ when p divides $|G|$.

Representation theory over fields of positive characteristic is called *modular representation theory.*

Example. Let $\phi, \psi \colon \mathbb{Z}_4 \to GL_2(\mathbb{C})$ be given by

$$\phi_{[k]} = A^k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k, \qquad \psi_{[k]} = B^{-k} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^k.$$

We can get a morphism from ϕ to ψ by starting with the identity matrix and averaging:

$$\begin{aligned} P &:= \frac{1}{4}\big(I + BA^{-1} + B^2A^{-2} + B^3A^{-3}\big) \\ &= \frac{1}{4}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \end{aligned}$$

which gives a morphism (in fact, an equivalence) from ϕ to ψ.

9. Decomposability of representations of finite groups

Proposition. *Every non-0 (finite dimensional, complex) representation of a finite group is either irreducible or decomposable.*

Proof. Consider a representation $\phi \colon G \to GL(V)$ which is not irreducible. Thus there exists a G-invariant subspace $W \leq V$ with $W \neq 0$ and $W \neq V$.

Therefore there is a linear map $T \colon V \to V$ such that $T|_W = \mathrm{id}_W$ and $T(V) \subseteq W$ (a "projection operator" onto the subspace W). To construct T explicitly, choose a basis $v_1, \dots, v_n$ of V so that $v_1, \dots, v_k$ is a basis of W (so $0 < k < n$). Let $W' = \mathbb{C}\{v_{k+1}, \dots, v_n\}$, the span of the remaining basis vectors. Then V is an internal direct sum of W and W' as a vector space (but perhaps not as a representation). We define T so that $T(v_i) = v_i$ if $i \leq k$, and $T(v_i) = 0$ if $i > k$. Note that $TT = T$.

Now average T to get $T' \colon V \to V$, where

$$T' := \frac{1}{|G|} \sum_{g \in G} \phi_g T \phi_{g^{-1}}.$$

Note that $T' \in \mathrm{Hom}_G(\phi, \phi)$, since this is the averaging trick.

We have that $T'(V) \subseteq W$, since $T(\phi_{g^{-1}}(v)) \in W$ for all $v \in V$, and thus $\phi_g T \phi_{g^{-1}}(v) \in W$ since W is an invariant subspace.

We have that $T'|_W = \mathrm{id}_W$, since for $w \in W$, $\phi_{g^{-1}}(w) \in W$, so $T\phi_{g^{-1}}(w) = \phi_{g^{-1}}(w)$, so

$$T'(w) = \frac{1}{|G|} \sum_{g \in G} \phi_g T \phi_{g^{-1}}(w) = \frac{1}{|G|} \sum_{g \in G} \phi_g \phi_{g^{-1}}(w) = w.$$

Since $T'(W) \subseteq W$ this implies $T'T' = T'$.

Now consider

$$W' := \ker(T') = \{\, x \in V \mid T'(x) = 0 \,\}.$$

Since $T' \in \mathrm{Hom}_G(\phi, \phi)$, this is an invariant subspace of V.

I claim that V is a direct sum of W and W'. Note that for all $x \in V$, we have

$$x = T'(x) + (x - T'(x)), \qquad T'(x) \in W, \quad x - T'(x) \in W'.$$

This is because $T'(I - T') = T' - T'T' = 0$. Also $W \cap W' = 0$, since for $x \in W \cap W'$ we have $x = T'(x) = 0$.

We have shown that ϕ is decomposable: it is a direct sum of subrepresentations $\phi|_W$ and $\phi|_{W'}$. □

10. Complete reducibility of representations of finite groups

Theorem (Maschke)**.** *Every finite dimensional representation of a finite group is completely reducible.*

Proof. Induction on dimension. That is, if ϕ is not 0 or irreducible, then $\phi \sim \phi_{(1)} \oplus \phi_{(2)}$ for non-0 subrepresentations $\phi_{(1)}$ and $\phi_{(2)}$, which each have strictly smaller dimension than ϕ, and so are completely reducible by induction. □

Remark. This theorem really does need G to be finite, and the characteristic of the field F to be 0.

For instance, consider

$$\phi\colon \mathbb{Z} \to GL_2(\mathbb{C}), \qquad \phi(k) := \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

This is a representation of the infinite group $\mathbb{Z}$. It is not irreducible, but also not decomposable. (Exercise: prove this.)

Similarly, for a prime p consider

$$\phi\colon \mathbb{Z}_p \to GL_2(\mathbb{Z}_p), \qquad \phi([k]) := \begin{bmatrix} [1] & [k] \\ [0] & [1] \end{bmatrix}.$$

This is a representation of $\mathbb{Z}_p$ in a vector space over $\mathbb{Z}_p$. It is not irreducible, but also not decomposable. (Exercise: prove this.)

Corollary. *If $\phi\colon G \to GL_n(\mathbb{C})$ is a matrix representation of a finite group, then there exists $T \in GL_n(\mathbb{C})$, such that*

$$T^{-1}\phi T = \begin{bmatrix} \phi^{(1)} & 0 & \cdots & 0 \\ 0 & \phi^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi^{(r)} \end{bmatrix}$$

where each $\phi^{(k)}\colon G \to GL_{n_k}(\mathbb{C})$ is an irreducible matrix representation.

Proof. Since the representation is completely reducible there are subspaces $V_1, \ldots, V_r$ of V such that (i) V is a direct sum of $V_1, \ldots, V_r$ and (ii) each $\phi|_{V_k}$ is irreducible. Choose a basis $v_1, \ldots, v_n$ so that $v_1, \ldots, v_{n_1}$ are a basis for V_1, $v_{n_1+1}, \ldots, v_{n_2}$ are a basis of V_2, etc. Then let $T = [v_1 \ \cdots \ v_n] \in GL_n(V)$. □

Example. For the standard representation $\rho\colon S_3 \to GL_3(\mathbb{C})$, we have

$$T^{-1}\rho_{(1\ 2)}T = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & 1 \end{array}\right], \qquad T^{-1}\rho_{(1\ 2\ 3)}T = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right], \qquad T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

11. Schur's lemma

This is the key fact for studying irreducible representations.

Proposition (Schur's lemma)**.** *Let $\phi\colon G \to GL(V)$ and $\psi\colon G \to GL(W)$ be irreducible representations, and $T \in \mathrm{Hom}_G(\phi, \psi)$. Then either $T = 0$ or T is an isomorphism. As a consequence:*

(1) *If $\phi \not\sim \psi$, then $T = 0$.*
(2) *If $\phi \sim \psi$, then $\mathrm{Hom}_G(\phi, \psi)$ is a 1-dimensional vector space.*

In particular, $\mathrm{Hom}_G(\phi, \phi) = \{\, \lambda I \mid \lambda \in \mathbb{C} \,\}$.

Proof. First, suppose $T \in \mathrm{Hom}_G(\phi, \psi)$ with $T \neq 0$. Then $\ker(T)$ and $T(V)$ are invariant subspaces, and $\ker(T) \neq V$ and $T(V) \neq 0$. But since ϕ is irreducible the only invariant subspaces are 0 and V, so $\ker(T) = 0$ and $T(V) = V$, so T is an isomorphism.

(1) If $\phi \not\sim \psi$, then T cannot be an isomorphism, so $T = 0$.

(2) First suppose $V = W$ and $\phi = \psi$, and consider any $T \in \mathrm{Hom}_G(\phi, \phi)$. Since T is a $\mathbb{C}$-linear map, it has an eigenvalue $\lambda \in \mathbb{C}$, so $T - \lambda I$ is not an isomorphism of vector spaces. But remember that $\mathrm{Hom}_G(\phi, \phi)$ is a vector space, so $T - \lambda I \in \mathrm{Hom}_G(\phi, \phi)$. Since all non-0 elements of $\mathrm{Hom}_G(\phi, \phi)$ are isomorphisms, we must have $T - \lambda I = 0$, i.e., $T = \lambda I$ for some λ.

Now if $\phi \sim \psi$, fix an equivalence $S \in \mathrm{Hom}_G(\psi, \phi)$. Then for any $T \in \mathrm{Hom}_G(\phi, \psi)$, we have $TS \in \mathrm{Hom}_G(\psi, \psi) = \{\, \lambda I \mid \lambda \in \mathbb{C} \,\}$, so $T = \lambda S^{-1}$ for some $\lambda \in \mathbb{C}$, i.e., $\mathrm{Hom}_G(\phi, \psi) = \{\, \lambda S^{-1} \mid \lambda \in \mathbb{C} \,\}$ which is 1-dimensional. □

12. 1-dimensional representations and abelian groups

Proposition. *If G is a finite abelian group, then every irreducible representation has dimension 1.*

Proof. Let $\phi\colon G \to GL(V)$ be a representation. Given $a \in G$ let $T = \phi_a\colon V \to V$. Then for $g \in G$,

$$\phi_g T = \phi_g \phi_a = \phi_{ga} = \phi_{ag} = \phi_a \phi_g = T\phi_g,$$

That is, because G is abelian, every ϕ_g is an element of $\mathrm{Hom}_G(\phi, \phi)$. By Schur's lemma, if ϕ is irreducible we must have $\phi_g = \lambda_g I$ for some $\lambda_g \in G$.

In particular, for any $v \in V$, the subspace $\mathbb{C}v$ is G-invariant, since $\phi_g(v) = \lambda_g v$. Thus ϕ can be irreducible iff $\dim V = 1$. □

In terms of matrix representations complete reducibility of representations of finite abelian groups has the following form.

Corollary. *If G is a finite abelian group and $\phi\colon G \to GL_n(\mathbb{C})$ a representation, then there exists $T \in GL_n(\mathbb{C})$ such that $T^{-1}\phi_g T$ is diagonal for all $g \in G$.*

Corollary. *If $A \in GL_n(\mathbb{C})$ has finite order, then A is diagonalizable.*

Proof. If $o(A) = n$, then there is a representation $\phi\colon \mathbb{Z}_n \to GL_n(\mathbb{C})$ with $\phi([1]) = A$. □

Given a group G, let $[G,G] := \langle ghg^{-1}h^{-1},\ g, h \in G \rangle \leq G$, called the **commutator subgroup**, i.e., the subgroup of G generated by all commutators of all elements. **commutator subgroup**

Exercise. $[G,G]$ is a normal subgroup of G, and the quotient group $G/[G,G]$ is abelian.

Exercise. Every homomorphism $G \to H$ to an abelian group H contains $[G,G]$ in its kernel, and so factors through a homomorphism $G/[G,G] \to H$.

Example. The commutator subgroup of Q_8 (quaternion group of order 8) is $\{\pm 1\}$. The quotient $Q_8/[Q_8, Q_8]$ is a Klein 4-group, i.e., isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition. *The 1-dimensional representations of a finite group G are all of the form $\phi \circ \pi$:*

$$G \xrightarrow{\pi} G/[G,G] \xrightarrow{\phi} \mathbb{C}^\times$$

where ϕ is a 1-dimensional representation of $G/[G,G]$.

Given an arbitrary representation $\phi\colon G \to GL(V)$ of some dimension d, and a 1-dimensional representation $\theta\colon G \to \mathbb{C}^\times$, we can define a new representation of dimension d, by

$$\theta\phi := \big(g \mapsto \theta(g)\phi(g)\big).$$

Exercise. If $\phi\colon G \to GL(V)$ is irreducible and $\theta\colon G \to \mathbb{C}^\times$ a homomorphism, then $\theta\phi$ is also irreducible.

13. Trace of a linear operator

Given a square matrix $A = (a_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ its **trace** is **trace**

$$\operatorname{Tr} A := \sum_{i=1}^n a_{ii} = \sum_{i,j=1}^n a_{ij}\delta_{ij}.$$

We have that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for any $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, since

$$\operatorname{Tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n\sum_{j=1}^n a_{ij}b_{ji}, \qquad \operatorname{Tr}(BA) = \sum_{i=1}^n (BA)_{ii} = \sum_{i=1}^n\sum_{j=1}^n b_{ij}a_{ji},$$

which are the same. Therefore $\operatorname{Tr}(PAP^{-1}) = \operatorname{Tr} A$ if P is invertible, since $\operatorname{Tr}((PA)P^{-1}) = \operatorname{Tr}(P^{-1}(PA))$.

Let $T\colon V \to V$ be a linear map, where $\dim V < \infty$. Then we define the **trace** of T as follows: Choose a basis $B = \{v_1, \dots, v_n\}$ of V, let $A = [T]_B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the matrix of T in this basis. Then define $\operatorname{Tr} T := \operatorname{Tr} A$. **trace of a linear operator**

This does not depend on the choice of basis: with respect to another basis B, we have $[T]_{B'} = P[T]_B P^{-1}$ for some invertible matrix P.

Actually, we can do a little better.

Proposition. *Let $T \in \operatorname{Hom}(V,V)$, $S \in \operatorname{Hom}(W,W)$, and suppose $U\colon V \to W$ is an isomorphism such that $UTU^{-1} = S$. Then $\operatorname{Tr}(T) = \operatorname{Tr}(S)$.*

Proof. Pick a basis $B = \{v_1, \dots, v_n\}$ of V. Then $B' = \{w_1, \dots, w_n\}$ with $w_k = Sv_k$ is a basis of W. We have $[S]_{B'} = [T]_B$, and the claim follows. □

Trace behaves well with respect to direct sums.

Proposition. *Let $T_k\colon V_k \to V_k$ be linear operators for $k = 1, \dots, r$, where $\dim V_k < \infty$. Let $V := V_1 \oplus \cdots \oplus V_r$ and $T := T_1 \oplus \cdots \oplus T_r\colon V \to V$. Then*

$$\operatorname{Tr}(T_1 \oplus \cdots \oplus T_r) = \operatorname{Tr}(T_1) + \cdots + \operatorname{Tr}(T_r).$$

Proof. Choose a basis B of V so that $A = [T]$ is a block matrix

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r, \end{bmatrix}$$

where $A_r = [T_r]$. □

Note that trace defines a linear map $\mathrm{Tr}\colon \mathrm{Hom}(V,V) \to \mathbb{C}$, i.e., $\mathrm{Tr}(S+T) = \mathrm{Tr}(S) + \mathrm{Tr}(T)$ and $\mathrm{Tr}(\lambda T) = \lambda \,\mathrm{Tr}(T)$.

Exercise. Here is a characterization of trace that doesn't use a choice of basis. For a finite dimensional V, show that $\mathrm{Tr}\colon \mathrm{Hom}(V,V) \to \mathbb{C}$ is the unique linear map such that (i) Tr vanishes on commutators, i.e., $\mathrm{Tr}(AB - BA) = 0$ for all $A, B \in \mathrm{Hom}(V,V)$, and (ii) $\mathrm{Tr}\, I = \dim V$. [Hint: Reduce to the case of matrices. Let E_{ij} be the matrix with 1 in entry (i,j), and 0 everywhere else. Then if $i \neq j$ show that E_{ij} and $E_{ii} - E_{jj}$ are commutators.]

Exercise. Compute the trace of $\mathrm{Rot}_u(\theta) \in SO(3)$, the matrix which describes rotation by angle θ through an axis passing through a unit vector u in $\mathbb{R}^3$. For which values of θ is this trace an integer? Compute the trace of $\mathrm{Rot}_u(\theta)\,\mathrm{Refl}_u \in O(3)$, where Refl_u is the matrix which describes reflection across the plane perpendicular to u. For which values of θ is this trace an integer?

14. Character of a representation

Given a (finite dimensional) representation $\phi\colon G \to GL(V)$, the **character** of ϕ is a function $\chi_\phi\colon G \to \mathbb{C}$ (not usually a homomorphism) defined by **character**

$$\chi_\phi(g) := \mathrm{Tr}(\phi_g).$$

In practice, you compute the character by choosing a basis of V, so converting ϕ into a matrix representation. If ϕ is a matrix representation, then

$$\chi_\phi(g) = \sum_{i=1}^{n} \phi_{ii}(g),$$

where $\phi_g = (\phi_{ij}(g)) \in GL_n(\mathbb{C})$, with matrix entries $\phi_{ij}(g) \in \mathbb{C}$.

The importance of characters is from the following.

Proposition. *If $\phi \sim \psi$, then $\chi_\phi = \chi_\psi$. That is, the character is an* equivalence invariant *of representations.*

Proof. If $T \in \mathrm{Hom}(\phi,\psi)$, then $\psi_g = T\phi_g T^{-1}$, and thus $\mathrm{Tr}(\psi_g) = \mathrm{Tr}(\phi_g)$. □

As we will show later, more is true: characters are a *complete equivalence invariant.* That is, $\phi \sim \psi$ iff $\chi_\phi = \chi_\psi$. Thus, classifying representations up to equivalence amounts to understanding the possible characters.

Example. If $\phi\colon G \to GL_1(\mathbb{C})$ is a 1-dimensional representation, then $\chi_\phi = \phi$. For this reason, it is typical to quietly identify 1-dimensional representations with their characters.

Proposition. *We have $\chi_\phi(e) = \dim V$ and $\chi_\phi(g^{-1}) = \overline{\chi_\phi(g)}$. Furthermore, $\chi_\phi(g)$ is a a finite sum of numbers of the form ζ^k, where $\zeta = e^{2\pi i/d}$ with $d = o(g)$, the order of g.*

Proof. We can assume WLOG that ϕ is a matrix representation $G \to GL_n(\mathbb{C})$. The first claim is clear, since $\mathrm{Tr}(I) = n$. For the second claim, note that if $A = \phi_g$ then $o(A)$ divides d, so it is diagonalizable, and in fact similar to $\mathrm{diag}(\lambda_1,\dots,\lambda_n)$, where the eigenvalues must satisfy $\lambda_k^d = 1$. This implies that $\lambda_k = \zeta^{j_k}$ for some $j_k \in \mathbb{Z}$, and that $\lambda_k^{-1} = \zeta^{-j_k} = \overline{\lambda_k}$. Thus

$$\mathrm{Tr}\, A = \lambda_1 + \cdots + \lambda_n = \zeta^{j_1} + \cdots + \zeta^{j_n}, \qquad \mathrm{Tr}\, A^{-1} = \overline{\lambda_1} + \cdots + \overline{\lambda_n} = \zeta^{-j_1} + \cdots + \zeta^{-j_n},$$

since A^{-1} will be similar to $\mathrm{diag}(\lambda_1^{-1},\dots,\lambda_n^{-1})$. □

Exercise. Show that if $o(g) = 2$, then $\chi_\phi(g) \in \mathbb{Z}$. Show that if g is conjugate to g^{-1} in the group G, then $\chi_\phi(g) \in \mathbb{R}$. Show that $\chi_\phi(g) = \chi_\phi(e)$ iff $g \in \ker(\phi)$. (Hint: diagonalize ϕ_g.)

Exercise. Show that $|\chi_\phi(g)| \leq |\chi_\phi(e)|$ for any $g \in G$. (Hint: use the triangle inequality $|z_1 + \cdots + z_d| \leq |z_1| + \cdots + |z_d|$.)

Proposition. *If $\phi = \phi^{(1)} \oplus \cdots \oplus \phi^{(r)}$ is a direct sum of representations, then*

$$\chi_\phi = \chi_{\phi^{(1)}} + \cdots + \chi_{\phi^{(r)}}.$$

Proof. Reduce to the case of a matrix representation with block form

$$\phi = \begin{bmatrix} \phi^{(1)} & 0 & \cdots & 0 \\ 0 & \phi^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi^{(r)} \end{bmatrix}.$$

□

Proposition. *If $\theta \colon G \to \mathbb{C}^\times$ and $\phi \colon G \to GL(V)$ are representations, then the character of $\theta\phi \colon G \to GL(V)$ is*

$$\chi_{\theta\phi}(g) = \theta(g)\chi_\phi(g).$$

Proof. Immediate from the fact that for $\lambda \in \mathbb{C}$, $\mathrm{Tr}(\lambda\phi_g) = \lambda \,\mathrm{Tr}(\phi_g)$. □

Exercise. Let $\phi \colon G \to GL_n(\mathbb{C})$ be a matrix representation. Define $\overline{\phi} \colon G \to GL_n(\mathbb{C})$ by

$$\overline{\phi}(g) := \overline{\phi(g)},$$

where for a matrix $A = (a_{ij})$, we let $\overline{A} = (\overline{a_{ij}})$. Show that $\overline{\phi}$ is a representation, and show that $\chi_{\overline{\phi}} = \overline{\chi_\phi}$. Also show that $\overline{\phi}$ is irreducible iff ϕ is irreducible.

Exercise. Suppose $c \in Z(G)$ is an element in the center of G, and let ϕ be an irreducible G-representation with character $\chi = \chi_\phi$ and dimension $d = \dim \phi$. Show that $|\chi(c)| = \chi(e)$, and that $\chi(cg) = \chi(c)\chi(g)/d$ for all $g \in G$.

15. Class functions

A **class function** is a function $f \colon G \to \mathbb{C}$ such that **class function**

$$f(g) = f(hgh^{-1}) \qquad \text{for all } g, h \in G.$$

That is, the value of f on g depends only on the conjugacy class of g.

Proposition. *Characters of representations are class functions.*

Proof.

$$\chi_\phi(hgh^{-1}) = \mathrm{Tr}(\phi_{hgh^{-1}}) = \mathrm{Tr}(\phi_h\phi_g\phi_h^{-1}) = \mathrm{Tr}(\phi_g) = \chi_\phi(g).$$

□

If a class function is the character of some representation, we just call it a "character". Thus, the set of characters is a subset of the set of class functions.

We write $L(G) = \{f \colon G \to \mathbb{C}\}$ for the set of all functions to complex numbers , and $L^c(G) \subseteq L(G)$ for the subset of class functions[2]. Note that $L(G)$ is a complex vector space, and $L^c(G)$ is a subspace of $L(G)$. We have that $\dim L(G) = |G|$, and $\dim L^c(G) =$ the number of conjugacy classes in G.

The vector spaces $L(G)$ and $L^c(G)$ are *inner product spaces.* Define a function $L(G) \times L(G) \to \mathbb{C}$, written "$(f_1, f_2) \mapsto \langle f_1,\, f_2\rangle$", by

$$\langle f_1,\, f_2\rangle := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

[2]Steinberg calls this set "$Z(L(G))$".

This has the properties

- $\langle \lambda_1 f_1 + \lambda_2 f_2,\, g\rangle = \lambda_1\langle f_1,\, g\rangle + \lambda_2\langle f_2,\, g\rangle$, for $\lambda_1, \lambda_2 \in \mathbb{C}$, $f_1, f_2, g \in L(G)$.
- $\langle g,\, f\rangle = \overline{\langle f,\, g\rangle}$, for $f, g \in L(G)$.
- $\langle f,\, f\rangle \in \mathbb{R}_{>0}$ if $f \in L(G)$ and $f \neq 0$.

That is, it is an **Hermitian inner product** for $L(G)$. (Note: $\langle f_1,\, f_2\rangle$ is $\mathbb{C}$-linear in the first variable, but is $\mathbb{C}$-*antilinear* in the second variable, i.e., $\langle f_1,\, \lambda f_2\rangle = \overline{\lambda}\langle f_1,\, f_2\rangle$.) **Hermitian inner product**

As a consequence, it makes sense to speak of an **orthonormal subset** of $L(G)$ or of $L^c(G)$: a list of elements $f_1, \dots, f_n$ such that $\langle f_i,\, f_j\rangle = \delta_{ij}$. It is an immediate consequence that any orthonormal subset is linearly independent. **orthonormal subset**

An **orthonormal basis** is a basis which is an orthonormal subset. Given an orthonormal basis $f_1, \dots, f_n$, we always have **orthonormal basis**

$$f = \sum_{k=1}^{n} \langle f,\, f_k\rangle\, f_k.$$

16. Orthogonality relations for characters

We are going to show the following, called the *orthogonality relation for characters.*

Theorem. *Let ϕ, ψ be irreducible representations of G. Then*

$$\langle \chi_\phi,\, \chi_\psi\rangle = \begin{cases} 1 & \text{if } \phi \sim \psi, \\ 0 & \text{if } \phi \not\sim \psi. \end{cases}$$

That is, the irreducible characters are an orthonormal subset of $L^c(G)$.

I will prove this soon. First, let's get lots of consequences.

Corollary. *There are at most s equivalence classes of irreducible representations of G, where $s =$ the number of conjugacy classes in G.*

Proof. If $\lambda^1, \dots, \lambda^r$ are a list of pairwise inequivalent irreducible representations, then by the orthogonality relation, the characters $\chi_{\lambda^1}, \dots, \chi_{\lambda^r}$ are an orthonormal subset of $L^c(G)$ (and in particular are pairwise distinct, since the representations are pairwise inequivalent). Thus $r \leq \dim L^c(G) = s$. □

(Soon, we will show that irreducible characters are a *basis* of $L^c(G)$, and thus that there are exactly s distinct irreducible characters.)

Proposition. *Let ϕ be a representation of G, and suppose $\phi = \phi^{(1)} \oplus \cdots \oplus \phi^{(r)}$, where each $\phi^{(k)}$ is irreducible. Then for any irreducible representation λ,*

$$\langle \chi_\lambda,\, \chi_\phi\rangle = \text{number of } k \in \{1, \dots, n\} \text{ such that } \phi^{(k)} \sim \lambda.$$

As a consequence, any two decompositions of ϕ as a direct sum of irreducible representations have the same number of irreducibles of each type.

Proof. Since $\chi_\phi = \sum_{k=1}^{r} \chi_\phi^{(k)}$, we have

$$\langle \chi_\phi,\, \chi_\lambda\rangle = \sum_{k=1}^{n} \langle \chi_{\phi^{(k)}},\, \chi_\lambda\rangle,$$

where each term $\langle \chi_{\phi^{(k)}},\, \chi_\lambda\rangle$ is either 1 or 0 depending on whether $\lambda \sim \phi^{(k)}$. This gives the first statement.

The second statement follows from the fact that the numbers $\langle \chi_\lambda,\, \chi_\phi\rangle$ do not depend on the choice of direct sum decomposition. □

For an irreducible representation λ, the number $\langle \chi_\lambda, \chi_\phi \rangle$ is called the **multiplicity** of λ in ϕ. **multiplicity**

Proposition. *If ϕ is a representation, and $\lambda^1, \dots, \lambda^s$ is a complete list of pairwise inequivalent irreducible representations of G, then*

$$\phi \sim m_1\lambda^1 \oplus \cdots \oplus m_s\lambda^s,$$

where $m_k = \langle \chi_\phi, \chi_{\lambda^k} \rangle$, and "$m\lambda$" is shorthand for "$\underbrace{\lambda \oplus \cdots \oplus \lambda}_{m \text{ copies}}$".

Proof. Since ϕ is completely reducible, there exists a decomposition

$$\phi \sim m_1\lambda^1 \oplus \cdots m_s\lambda^s$$

with $m_k \geq 0$. Then

$$\langle \chi_\phi, \chi_{\lambda^k} \rangle = \langle \sum_{i=1}^s m_i \chi_{\lambda^i}, \chi_{\lambda^k} \rangle = \sum_{i=1}^s m_i \langle \chi_{\lambda^i}, \chi_{\lambda^k} \rangle = m_k.$$

□

Corollary. *If ϕ and ψ are representations, then $\chi_\phi = \chi_\psi$ iff $\phi \sim \psi$. That is, characters are a complete isomorphism invariant of (finite dimensional) representations of (finite) groups.*

Proof. Immeditate from the preceeding proposition. □

Corollary. *If $\phi \sim m_1\lambda_1 \oplus \cdots \oplus m_s\lambda_s$, where $\lambda_1, \dots, \lambda_s$ are pairwise inequivalent irreducibles, then*

$$\langle \chi_\phi, \chi_\phi \rangle = \sum_{k=1}^s m_k^2.$$

In particular, ϕ is irreducible iff $\langle \chi_\phi, \chi_\phi \rangle = 1$.

Proof. Compute

$$\begin{aligned}\langle \chi_\phi, \chi_\phi \rangle &= \langle \sum_{i=1}^r m_i \chi_{\lambda_i}, \sum_{j=1}^r m_j \chi_{\lambda_j} \rangle \\ &= \sum_{i=1}^r \sum_{j=1}^r m_i m_j \langle \chi_{\lambda_i}, \chi_{\lambda_j} \rangle \\ &= \sum_{k=1}^r m_k^2,\end{aligned}$$

using the orthogonality relation for characters. This $= 1$ iff $m_k = 1$ for exactly one value of k, and all other $m_i = 0$. □

Warning. it is not true that $f \in L^c(G)$ is such that $\langle f, f \rangle = 1$, then f is the character of some irreducible representation. You need to know that f is a character in order to conclude this.

17. Character tables for $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

Write a for a generator of $\mathbb{Z}_4$. The following table presents all the irreducible characters of $\mathbb{Z}_4$.

	e	a	a^2	a^3
χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_3	1	-1	1	-1
χ_4	1	$-i$	-1	$-i$

All these characters are homomorphisms $\mathbb{Z}_4 \to \mathbb{C}^\times$, which are all the irreducible representations since G is abelian.

You can check explicitly that the orthogonality relations $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ hold. Using

$$\langle \chi_i, \chi_j \rangle = \frac{1}{4}\big[\chi_i(e)\overline{\chi_j(e)} + \chi_i(a)\overline{\chi_j(a)} + \chi_i(a^2)\overline{\chi_j}(a^2) + \chi_i(a^3)\overline{\chi_j(a^3)}\big],$$

we get for instance

$$\langle \chi_2, \chi_2 \rangle = \frac{1}{4}\big[(1)(1) + (i)(-i) + (-1)(-1) + (-i)(i)\big] = 1$$

and

$$\langle \chi_2, \chi_4 \rangle = \frac{1}{4}\big[(1)(1) + (i)(i) + (-1)(-1) + (-i)(-i)\big] = 0.$$

Here is the character table for $\mathbb{Z}_2 \times \mathbb{Z}_2$, in terms of order 2 generators a and b.

	e	a	b	ab
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

18. Character table for S_3

Here are some characters.

- There is a trivial representation $S_3 \to \mathbb{C}^\times$, which is also a irreducible character, which we write as χ_1, so $\chi_1(g) = 1$ for all g.
- The sign homomorphism $S_3 \to \mathbb{C}^\times$ sends even permutations to 1 and odd permutations to -1. This is a representation, and also a irreducible character, which we write as χ_2.
- We have the standard representation $\rho\colon S_3 \to GL_3(\mathbb{C})$. It is easy to compute its character on representatives of conjugacy classes:

  $$\chi_\rho(e) = 3, \qquad \chi_\rho((1\ 2)) = 1, \qquad \chi_\rho((1\ 2\ 3)) = 0.$$

  We have

  $$\begin{aligned}\langle \chi_\rho, \chi_\rho \rangle &= \frac{1}{S_3}\sum_{g\in S_3} \chi_\rho(g)\overline{\chi_\rho(g)} \\ &= \frac{1}{6}\big[1(3\cdot 3) + 3(1\cdot 1) + 2(0\cdot 0)\big] \\ &= 2.\end{aligned}$$

  Thus, ρ is not an irreducible representation. The only way to write 2 as a sum of positive squares is $2 = 1^2 + 1^2$, so $\rho \sim \lambda \oplus \lambda'$ for two inequivalent irreducible representations λ, λ'.

  We can compute

  $$\langle \chi_1, \chi_\rho \rangle = \frac{1}{6}\big[1(1\cdot 3) + 3(1\cdot 1) + 2(1\cdot 0)\big] = 1$$

and
$$\langle \chi_2, \chi_\rho \rangle = \frac{1}{6}\big[1(1\cdot 3) + 3(-1\cdot 1) + 2(1\cdot 0)\big] = 0.$$
So $\rho \sim \chi_1 \oplus \psi$ for some other irreducible ψ.

- Since $\chi_\rho = \chi_1 + \chi_\psi$, we must have
$$\chi_\psi(e) = 2, \qquad \chi_\psi((1\ 2)) = 0, \qquad \chi_\psi((1\ 2\ 3)) = -1.$$

We obtain the "character table" of S_3, with $\chi_3 = \chi_\psi$:

6	1	3	2
	e	$(1\ 2)$	$(1\ 2\ 3)$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

I've written the sizes of the conjugacy classes above each representative, since these numbers are needed to compute the inner product of characters.

Note: we know that since the sign representation is 1-dimensional, if χ is an irreducible character so is $\chi_2\chi$. Since $\chi_2\chi_3$ is an irreducible character of dimension 2, and since we have accounted for all irreducible characters, we must have $\chi_2\chi_3 = \chi_3$. This gives another proof that $\chi_3((1\ 2)) = 0$, since $\chi_2((1\ 2)) = -1$.

Finally, note that ψ must be equivalent to the representation on the 2-dimensional invariant subspace of ρ that we described earlier. That earlier calculation thus gives yet another way to compute χ_3.

19. Proof of the orthogonality relations, part 1

Let $\phi\colon G \to GL(V)$ be an irreducible representation, and let $\chi = \chi_\phi$ be its character. I want to show $\langle \chi, \chi \rangle = 1$.

WLOG we can assume ϕ is a matrix representation, so we can write $\phi_g = (\phi_{ij}(g)) \in GL_n(\mathbb{C})$. Remember that $\overline{\chi(g)} = \chi(g^{-1})$, and compute:
$$\begin{aligned}\langle \chi, \chi \rangle &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\chi(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{i,j} \phi_{ii}(g)\phi_{jj}(g^{-1}).\end{aligned}$$
The second sum is over the indices $i = 1, \dots, n$ and $j = 1, \dots, n$. The idea is to switch the order of summation, and find a formula for $(1/|G|)\sum_{g\in G}\phi_{ii}(g)\phi_{jj}(g^{-1})$ for a fixed i and j.

We will use the "averaging trick" combined with Schur's lemma.

Lemma. *Let $\phi\colon G \to GL(V)$ be an irreducible representation of dimension n, and let $T\colon V \to V$ be a linear map. Define $T'\colon V \to V$ by*
$$T' := \frac{1}{|G|}\sum_{g\in G}\phi_g T \phi_g^{-1}.$$
Then $T' = \lambda I$ where $\lambda = (1/n)\operatorname{Tr}(T)$.

Proof. This is the "averaging" trick, so $T' \in \operatorname{Hom}_G(\phi, \phi)$. Since ϕ irreducible, Schur's lemma says $T' = \lambda I$ for some $\lambda \in \mathbb{C}$. To compute λ take the trace:
$$\operatorname{Tr}(T') = \frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}(\phi_g T\phi_g^{-1}) = \operatorname{Tr}(T),$$
so $\operatorname{Tr}(T) = \operatorname{Tr}(T') = \operatorname{Tr}(\lambda I) = n\lambda$ gives the claim. □

Assume now that $\phi\colon G \to GL_n(\mathbb{C})$ is a matrix representation. We represent an arbitrary $T \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ by an arbitrary matrix $X = (x_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

Let $X' = (x'_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the matrix representing T', so

$$X' := \frac{1}{|G|}\sum_{g\in G} \phi_g X \phi_{g^{-1}}.$$

By the lemma $X' = \lambda I$ with $\lambda = (1/n)\,\mathrm{Tr}\,X$.

I'm going to write the matrix entries of X' in two different ways. I'll use the Kronecker delta, defined by $I = (\delta_{ij})$. The averaging formula becomes:

$$x'_{ij} = \frac{1}{|G|}\sum_{g\in G}\sum_{u,v} \phi_{iu}(g) x_{uv} \phi_{vj}(g^{-1}),$$

where $1 \le u, v \le n$. Since $X' = (1/n)\,\mathrm{Tr}(X) I$, we can also write

$$x'_{ij} = \frac{1}{n}\delta_{ij}\sum_k x_{kk} = \frac{1}{n}\sum_{u,v}\delta_{ij} x_{uv}\delta_{uv}.$$

Putting this in one equation gives

$$0 = \sum_{u,v}\left[\frac{1}{n}\delta_{ij}\delta_{uv} - \frac{1}{|G|}\sum_{g\in G}\phi_{iu}(g)\phi_{vj}(g^{-1})\right] x_{uv}.$$

Remember that the matrix X is *arbitrary*. So if we set $x_{uv} = 1$ and all other entries of X to 0, we get a **Schur orthogonality relation**:

Schur orthogonality relation

$$\frac{1}{n}\delta_{ij}\delta_{uv} = \frac{1}{|G|}\sum_{g\in G}\phi_{iu}(g)\phi_{vj}(g^{-1}) \qquad \text{for all } 1 \le i, j, u, v \le n.$$

In particular, taking $u = i$ and $v = j$ gives

$$\frac{1}{n}\delta_{ij} = \frac{1}{|G|}\sum_{g\in G}\phi_{ii}(g)\phi_{jj}(g^{-1}) \qquad \text{for all } 1 \le i, j \le n.$$

Summing over all i and j gives

$$\begin{aligned}\langle \chi, \chi\rangle &= \sum_{i,j}\frac{1}{|G|}\sum_{g\in G}\phi_{ii}(g)\phi_{jj}(g^{-1}) \\ &= \sum_{i,j}\frac{1}{n}\delta_{ij} = 1.\end{aligned}$$

This is what we wanted.

20. Proof of the orthogonality relations, part 2

Let $\phi\colon G \to GL(V)$ and $\phi\colon G \to GL(W)$ be irreducible representations which are not equivalent. I want to show $\langle \chi_\psi, \chi_\phi\rangle = 0$. The proof will be almost the same as before.

WLOG we can assume ϕ and ψ are matrix representation, so we can write $\phi_g = (\phi_{ij}(g)) \in GL_n(\mathbb{C})$ and $\psi_g = (\psi_{ij}(g)) \in GL_m(\mathbb{C})$. Remember that $\overline{\chi_\phi(g)} = \chi_\phi(g^{-1})$, and compute:

$$\begin{aligned}\langle \chi_\psi, \chi_\phi\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_\psi(g)\chi_\phi(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{i,j}\psi_{ii}(g)\phi_{jj}(g^{-1}).\end{aligned}$$

Again, we will switch the order of summation, and find a formula for $(1/|G|)\sum_{g\in G}\psi_{ii}(g)\phi_{jj}(g^{-1})$ for a fixed i and j.

We will use the "averaging trick" combined with Schur's lemma.

Lemma. *Let $\phi\colon G\to GL(V)$ be an irreducible representation, and let $T\colon V\to V$ be a linear map. Define $T'\colon V\to V$ by*

$$T' := \frac{1}{|G|}\sum_{g\in G}\phi_g T\phi_g^{-1}.$$

Then $T'=0$.

Proof. This is the "averaging" trick, so $T'\in \mathrm{Hom}_G(\psi,\phi)$, and thus $T'=0$ by Schur's lemma since ϕ and ψ are irreducible and not equivalent. □

Assume now that $\phi\colon G\to GL_n(\mathbb{C})$ and $\psi\colon G\to GL_m(\mathbb{C})$ are matrix representations. We represent an arbitrary $T\in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^m)$ by an arbitrary matrix $X=(x_{ij})\in \mathrm{Mat}_{m\times n}(\mathbb{C})$.

Let $X'=(x'_{ij})\in \mathrm{Mat}_{m\times n}(\mathbb{C})$ be the matrix representing T', so

$$X' := \frac{1}{|G|}\sum_{g\in G}\psi_g X\phi_{g^{-1}}.$$

By the lemma $X'=0$.

The matrix entries of X' are

$$x'_{ij} = \frac{1}{|G|}\sum_{g\in G}\sum_{u,v}\psi_{iu}(g)x_{uv}\phi_{vj}(g^{-1}),$$

where $1\le u\le m$ and $1\le v\le n$. Since $X'=0$, we get

$$0=\sum_{u,v}\left[\frac{1}{|G|}\sum_{g\in G}\psi_{iu}(g)\phi_{vj}(g^{-1})\right]x_{uv}.$$

Since X is arbitrary, we can set $x_{uv}=1$ and all other entries of X to 0, and get another **Schur orthogonality relation**:

Schur orthogonality relation

$$0=\frac{1}{|G|}\sum_{g\in G}\psi_{iu}(g)\phi_{vj}(g^{-1}) \qquad \text{for all } 1\le i,u\le m,\ 1\le v,j\le n.$$

In particular, taking $u=i$ and $v=j$ gives

$$0=\frac{1}{|G|}\sum_{g\in G}\psi_{ii}(g)\phi_{jj}(g^{-1}) \qquad \text{for all } 1\le i\le m,\ 1\le j\le n.$$

Summing over all i and j gives

$$\langle \chi_\psi,\,\chi_\phi\rangle = \sum_{i,j}\frac{1}{|G|}\sum_{g\in G}\psi_{ii}(g)\phi_{jj}(g^{-1})=0.$$

This is what we wanted.

21. The regular representation

Any group action $G\to \mathrm{Sym}(X)$ on a finite set X can be upgraded to a representation, which I will call a **permutation representation**. To do this, let V be a vector space with a basis B which is in bijective correspondence in X. I'll write $u_x\in B$ for the basis element corresponding to $x\in X$. I'm going to write $\mathbb{C}X$ for this vector space, which has dimension $d=|X|$.

permutation representation

Now let

$$\rho\colon G\to GL(\mathbb{C}X)$$

be defined by

$$\rho_g(u_x) := u_{gx}.$$

This is a representation of G of dimension $d = |X|$. There is an easy formula for its character.

Proposition. *If χ is the character of the permutation representation associated to the action by G on X, then*

$$\chi(g) = |\mathrm{Fix}(g)|, \qquad \textit{where} \qquad \mathrm{Fix}(g) := \{\, x \in X \mid gx = x \,\}.$$

In particular, the characters of permutation representations are non-negative integer valued.

Proof. List the elements of X as $x_1, \dots, x_d$, and write u_i for u_{x_i}. Let $A = [\phi_g]_B$, the matrix of ϕ_g with respect to the basis B. Then A is a permutation matrix. The (i, i) entry is either 0 (if $gx_i \neq x_i$), or is 1 (if $gx_i = x_i$). □

Example (The standard representation of S_n). The standard representation $\rho\colon S_n \to GL_d(\mathbb{C})$ is an example of a permutation representation. Its character is given as follows: if $g \in S_n$ has cycle type $1^{r_1} 2^{r_2} \cdots n^{r_n}$ (i.e., a product of r_1 1-cycles, r_2 2-cycles, etc., pairwise disjoint, with $r_1 + \cdots + r_n = d$), then $\chi_\rho(g) = r_1$.

The **regular representation** of G is the one associated to the "left regular action" by G on $X = G$, defined by $x \mapsto gx$. It has dimension $n = |G|$. Explicitly, it is a homomorphism $L\colon G \to GL(\mathbb{C}G)$ given by **regular representation**

$$L_g(u_h) := u_{gh}.$$

Its character is given by

$$\chi_L(g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

It turns out that *every* irreducible representation of G is a subrepresentation of the regular representation. In fact, we have the following.

Theorem. *Let L be the regular representation of G, and let $\lambda_1, \dots, \lambda_s$ be a complete list of pairwise inequivalent irreducible representations of G, and write $d_k = \dim \lambda_k > 0$. Then*

$$L \sim d_1\lambda_1 \oplus \cdots \oplus d_s\lambda_s.$$

Furthermore,

$$|G| = d_1^2 + \cdots + d_s^2.$$

Proof. Let $\chi_k = \chi_{\lambda_k}$. Compute:

$$\langle \chi_L, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g)\overline{\chi_k(g)} = \overline{\chi_k(e)} = d_k.$$

So by the orthogonality relation, d_k = the multiplicity of λ_k in ρ. Since irreducible representations have positive dimension, $d_k > 0$, there is at least one copy of λ_k in ρ.

Finally, we can evaluate χ_L at e:

$$|G| = \chi_L(e) = \sum_k d_k \chi_k(e) = \sum_k d_k^2.$$

□

22. The number of irreducible representations

We have already shown that the number of irreducible representations (up to equivalence), is bounded above by the number of conjugacy classes in G. We will now show these numbers are equal.

Lemma. *Let $\phi\colon G \to GL(V)$ be a representation, and let $f \in L^c(G)$ be a class function. Define a function $T_f\colon V \to V$ by*

$$T_f := \frac{1}{|G|} \sum_{g\in G} \overline{f(g)}\phi_g.$$

Then $T_f \in \mathrm{Hom}_G(\phi, \phi)$.

Proof. For $a \in G$ we have

$$\begin{aligned} \phi_a T_f \phi_{a^{-1}} &= \frac{1}{|G|} \sum_{g\in G} \overline{f(g)}\phi_{aga^{-1}} \\ &= \frac{1}{|G|} \sum_{h\in G} \overline{f(a^{-1}ha)}\phi_h \qquad \text{where } h = aga^{-1}, \\ &= \frac{1}{|G|} \sum_{h\in G} \overline{f(h)}\phi_h = T_f, \end{aligned}$$

since f is a class function so $f(a^{-1}ha) = f(h)$. □

We get a different operator for each representation ϕ, so I might write T_f^ϕ instead of just T_f to emphasize this.

Corollary. *Let ϕ be an irreducible representation of dimension d and character χ. Then*

$$T_f = \sum_{g\in G} \overline{f(g)}\phi_g = \lambda I, \qquad \lambda = \frac{1}{d}\langle \chi,\, f\rangle.$$

Proof. By Schur's lemma, $T' = \lambda I$ for some λ. To compute λ we compute the trace of T_f:

$$\mathrm{Tr}(T_f) = \frac{1}{|G|} \sum_{g\in G} \overline{f(g)}\chi(g) = \langle \chi,\, f\rangle.$$

Since $\mathrm{Tr}(\lambda I) = \lambda d$, this gives the claim. □

Theorem. *Let $\lambda_1, \dots, \lambda_s$ be a complete set of pairwise inequivalent irreducible rerpresentations of G, and write $\chi_k = \chi_{\lambda_k}$ for the character of λ_k. Then $\chi_1, \dots, \chi_s$ are an orthonormal basis of $L^c(G)$, and thus $s =$ the number of conjugacy classes in G.*

Proof. We already know that $\langle \chi_i,\, \chi_j\rangle = \delta_{ij}$, so it suffices to show that if $f \in L^c(G)$ is such that $\langle \chi_k,\, f\rangle = 0$ for all k, then $f = 0$.

Let $\phi\colon G \to GL(V)$ be any representation. By complete reducibility, it has the form $\phi = \phi^{(1)} \oplus \cdots \oplus \phi^{(d)}$ for some irreducible subrepresetations $\phi^{(k)}\colon G \to GL(V_k)$, where $V_k \leq V$ is a subspace. Consider the operator $T_f^\phi \in \mathrm{Hom}_G(\phi, \phi)$ as defined above by $T_f^\phi = (1/|G|)\sum_g \overline{f(g)}\phi_g$. Note that this formula implies that $T_f(V_k) \subseteq V_k$, and thus that T_f^ϕ restricts to a morphism $T_f|V_k \in \mathrm{Hom}_G(\phi^{(k)}, \phi^{(k)})$, and from its formula we see that in fact $T_f|V_k = T_f^{\phi^{(k)}}$. By the above Corollary and the hypothesis that f is orthogonal to all irreducible characters, we must have $T_f|V_k = 0$ for all k, and therefore that $T_f = 0$. Thus, we have proved that the operator $T_f^\phi \in \mathrm{Hom}_G(\phi, \phi)$ is the zero map for *every* representation ϕ.

Let L be the regular representation, and evaluate T_f^L at the element $u_e \in \mathbb{C}G$:

$$T_f^L(u_e) = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} u_g.$$

Since the u_g are a basis of the regular representation, $T_f^L = 0$ implies $\overline{f(g)} = 0$ for all $g \in G$. Thus $f = 0$ as desired.

□

23. Character table for D_4

Let D_4 be the dihedral group of order 8, generated by elements r, j with $r^4 = e = j^2$ and $rj = jr^{-1}$.

8	1	2	1	2	2
	e	r	r^2	j	jr
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

The first four characters are homomorphisms $G \to \mathbb{C}^\times$. If $G = D_4$, then $[G, G] = \{e, r^2\}$, so $G/[G, G]$ is a Klein 4-group.

We can deduce the fifth character using the orthogonality relations and the fact that its dimension $\chi_5(e)$ must be 2, since $8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$. (For instance, we must have $\langle \chi_1 + \chi_2 + \chi_3 + \chi_4, \chi_5 \rangle = 0$, from which you can read off that $\chi_5(r^2) = -\chi_5(e)$. Then the fact that $\langle \chi_5, \chi_5 \rangle = 1$ already implies that χ_5 vanishes on $G \setminus \{e, r^2\}$. Alternately, we can use the fact that we must have $\chi_k \chi_5 = \chi_5$ for $k = 2, 3, 4$ to deduce this.)

The character χ_5 is that of the "obvious" real representation $\phi\colon G \to GL_2(\mathbb{R})$, defined by

$$\phi(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \phi(j) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since $GL_2(\mathbb{R}) \leq GL_2(\mathbb{C})$, this also gives a complex representation.

24. Second orthogonality relations

Let $\lambda_1, \dots, \lambda_s$ be a complete list of inequivalent irreducibles, with characters $\chi_1, \dots, \chi_s$ and let $g_1, \dots, g_s$ be a list of representatives of conjugacy classes in G. Then we can form the character table of G, which is really an $s \times s$ complex matrix:

	g_1	$\cdots$	g_s
χ_1	$\chi_1(g_1)$	$\cdots$	$\chi_1(g_s)$
$\vdots$	$\vdots$		$\vdots$
χ_s	$\chi_s(g_1)$	$\cdots$	$\chi_s(g_s)$

Recall that a matrix $U \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is **unitary** if its rows are an orthonormal basis of $\mathbb{C}^n$ (using the usual Hermitian inner product on $\mathbb{C}^n$). This is equivalent to saying $UU^* = I$, where $U^* = \overline{U}^\top$, which is equivalent to $U^*U = I$, which is equivalent to saying the columns are an orthornormal basis of $\mathbb{C}^n$. The character table is *almost* a unitary matrix. We write $\mathrm{Cl}(g) := \{\, hgh^{-1} \mid h \in G \,\}$ for the conjugacy class of g, and $\mathrm{Cent}(g) := \{\, h \in G \mid hgh^{-1} = g \,\}$ for the centralizer subgroup of g. **unitary**

Lemma. *The matrix* $U = (u_{ij}) \in \mathrm{Mat}_{s\times s}(\mathbb{C})$ *defined by*

$$u_{ij} = \chi_i(g_j)/\sqrt{c_j}, \qquad c_j = |\mathrm{Cent}(g_j)| = |G| \,/\, |\mathrm{Cl}(g_j)|$$

is a unitary matrix.

Proof. The inner product of the ith and jth rows of U is

$$\begin{aligned}\sum_{k=1}^{s} u_{ik}\overline{u_{jk}} &= \sum_{k=1}^{s} \frac{1}{c_k}\chi_i(g_k)\overline{\chi_j(g_k)} \\ &= \sum_{k=1}^{s} \frac{|\mathrm{Cl}(g_k)|}{|G|}\chi_i(g_k)\overline{\chi_j(g_k)} \\ &= \frac{1}{|G|}\sum_{k=1}^{s}\sum_{g\in \mathrm{Cl}(g_k)} \chi_i(g)\overline{\chi_j(g)} \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \langle \chi_i,\, \chi_j\rangle = \delta_{ij}.\end{aligned}$$

□

Theorem (Second orthogonality relations)**.** *Let $\chi_1, \dots, \chi_s$ be a complete list of pairwise distinct irreducible characters. For any $g \in G$ we have*

$$\sum_{k=1}^{s} \chi_k(g)\overline{\chi_k(g)} = |\mathrm{Cent}(g)| = |G| \,/\, |\mathrm{Cl}(g)|\,,$$

and for any $g, h \in G$ which are not conjugate to each other, we have

$$\sum_{k=1}^{s} \chi_k(g)\overline{\chi_k(h)} = 0.$$

In particular, columns of the character table are pairwise orthogonal.

Proof. This is just the fact that the columns of U are also an orthonormal basis, so $\sum_{k=1}^{s} u_{ki}\overline{u_{kj}} = \delta_{ij}$. □

Example (Character table for S_4). It looks like this.

24	1	6	3	6	8
	e	(1 2)	(1 2)(3 4)	(1 2 3 4)	(1 2 3)
χ_1	1	1	1	1	1
χ_2	1	−1	1	−1	1
χ_3	2	0	2	0	−1
χ_4	3	1	−1	−1	0
χ_5	3	−1	−1	1	0
	24	4	8	4	3

I have put the numbers $|\mathrm{Cent}(g)| = |G| \,/\, |\mathrm{Cl}(g)|$ along the bottom row, to make it easier to verify the second orthogonality relations.

Exercise. Prove this character table.

Exercise. There is a 3-dimenional real representation of S_4, coming from the fact that S_4 is isomorphic to the subgroup $G \le SO(3) \le GL_3(\mathbb{R}) \le GL_3(\mathbb{C})$ of rotational symmetries of the cube. Determine the character of this representation, and identify its decomposition as a direct sum of irreducible complex representations.

25. Frobenius divisibility, part 1

Here is one more fact about the dimensions of irreducible representations, whose proof is a bit more subtle than what we have seen so far.

Theorem (Frobenius)**.** *Let ϕ be an irreducible representation of G. Then $d = \dim \phi$ divides $n = |G|$.*

Consider any representation $\phi\colon G \to GL(V)$. For any $x \in G$, define the linear operator

$$T_x := \sum_{g \in \mathrm{Cl}(x)} \phi_g.$$

Note that this only depends on the conjugacy class of x. We have $\phi_a T_x \phi_{a^{-1}} = T_x$ for any $a \in G$, so $T_x \in \mathrm{Hom}_G(\phi, \phi)$.

Now suppose ϕ is irreducible, so Schur's lemma says that $T_x = \lambda_x I$ for some $\lambda_x \in \mathbb{C}$. We can actually compute λ_x by taking traces:

$$\lambda_x = \frac{|\mathrm{Cl}(x)|}{d} \chi(x)$$

where χ is the character of ϕ. Note that $\lambda_e = 1$.

Consider the linear map $\sum_{g \in G} \chi(g^{-1})\phi_g = \sum_{x_i} \chi(x_i^{-1}) T_{x_i}$, where in the second formula we sum over a list $x_1, \dots, x_s$ of representatives of the distinct conjugacy classes in G. By taking the trace of this, and using the orthogonality relation $\langle \chi, \chi \rangle = 1$, we get an identity

$$\frac{n}{d} = \sum_{x_i} \chi(x_i^{-1}) \lambda_{x_i}.$$

Let $R =$ the abelian subgroup of $\mathbb{C}$ generated under addition by the finite set of elements

$$\zeta^k \lambda_x, \qquad 0 \le k < n, \quad x \in G,$$

where $\zeta = e^{2\pi i/n}$. Since each $\chi(g)$ is a sum of n-th roots of unity, we have from the above identity that $n/d \in R$.

The theorem is an immediate consequence of the following.

Proposition. $R \cap \mathbb{Q} = \mathbb{Z}$.

Proof of Frobenius divisibility. We have shown that the rational number n/d is in R, and therefore by the proposition is an integer. □

We will prove this proposition in the next section.

Example (Character table of S_5). Here it is.

120	1	10	15	20	30	24	20
	e	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)	(1 2 3 4 5)	(1 2 3)(4 5)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	-1
χ_3	4	2	0	1	0	-1	-1
χ_4	4	-2	0	1	0	-1	1
χ_5	5	-1	1	-1	1	0	-1
χ_6	5	1	1	-1	-1	0	1
χ_7	6	0	-2	0	0	1	0
	120	12	8	6	4	5	6

The abelian characters are χ_1 and χ_2, where χ_2 is the sign representation. The sum $\chi_1 + \chi_3 = \chi_\rho$, where ρ is the standard 5-dimensional representation, can be computed explicitly.

Exercise. Explain how to prove the rest of the character table.

(It is possible to do this just using general facts about characters of irreducible representations that we have proved, including Frobenius divisibility, applied to $G = S_5$, together with the existence of the standard representation.)

(In addition, you could introduce some new representations whose characters you can compute. For instance, for every subgroup $H \le S_5$ there is a permutation representation $\mathbb{C}(S_5/H)$. These won't usually be irreducible, but they can contain new irreducible representations. For instance $H = N(\langle(1\ 2\ 3\ 4\ 5)\rangle)$, which gives a 6 dimensional rerpresentation, or $H = N(\langle(1\ 2\ 3)\rangle)$ or $H = N(\langle(1\ 2)\rangle)$, which give 10 dimensional representations.)

26. Frobenius divisibility, part 2

Recall that for an irreducible representation ϕ, we defined $R \le \mathbb{C}$ to be the subgroup generated by the set of numbers of the form $\zeta^k\lambda_g$, $0 \le k < n$ and $g \in G$, where $\zeta = e^{2\pi i/n}$ and λ_g is defined by

$$T_g = \sum_{x \in \mathrm{Cl}(g)} \phi_x = \lambda_g I.$$

Our goal is to show $R \cap \mathbb{Q} = \mathbb{Z}$.

Lemma. *R is a subring of $\mathbb{C}$, and $1 \in R$.*

Proof. By definition $(R,+)$ is a subgroup of $(\mathbb{C},+)$. We have that $\lambda_e = 1 \in R$. To show that R is a subring, it suffices to show that $\zeta^i\lambda_g\,\zeta^j\lambda_h \in R$ for all $0 \le i,j < n$ and $g,h \in G$. Since $\zeta^n = 1$, we have $\zeta^i\zeta^j = \zeta^k$ for some $0 \le k < n$. So it suffices to prove a formula of the form

$$\lambda_g\lambda_h = \sum_{x_i} m_{x_i}\lambda_{x_i}$$

where the $m_{x_i} \in \mathbb{Z}$.

Since $T_g = \lambda_g I$, it suffices to prove that

$$T_gT_h = \sum_{x_i} m_{x_i}T_{x_i},$$

for some $m_i \in \mathbb{Z}$. We have that

$$T_gT_h = \sum_{u\in \mathrm{Cl}(g)}\sum_{v \in \mathrm{Cl}(h)} \phi_{uv} = \sum_{x\in G} m_x\phi_x,$$

where m_x is the size of the set $M_x := \{\, (u,v) \in \mathrm{Cl}(g)\times\mathrm{Cl}(h) \mid uv = x\,\}$. For any $y \in G$, $(u,v) \mapsto (yuy^{-1}, yvy^{-1})$ defines a bijection $M_x \to M_{yxy^{-1}}$. Thus $m_x = m_{x'}$ whenever x and x' are conjugate, so

$$T_gT_h = \sum_{x\in G} m_x\phi_x = \sum_{x_i} m_{x_i}T_{x_i}.$$

□

Recall that the underlying abelian group $(R,+)$ of R is finitely generated, by definition. Thus the claim that $R \cap \mathbb{Q} = \mathbb{Z}$ follows from the following statement.

Lemma. *Let R be a subring of $\mathbb{C}$ containing 1, such that the underlying abelian group of R is finitely generated. Then $R \cap \mathbb{Q} = \mathbb{Z}$.*

Proof. Since $1 \in R$ it is clear that $\mathbb{Z} \subseteq R \cap \mathbb{Q}$.

Note that $(R,+)$, in addition to being finitely generated, is torsion free, since $nr \ne 0$ when $n \in \mathbb{Z} \smallsetminus \{0\}$ and $r \in \mathbb{C} \smallsetminus \{0\}$, since $\mathbb{C}$ is a field of characteristic 0. So by the classification of finitely generated abelian groups, $(R,+)$ is isomorphic to $\mathbb{Z}^m$ for some $m \ge 1$. So we can choose a $\mathbb{Z}$-basis $B = \{e_1,\dots,e_m\}$ for R.

Because R is a ring, for any $\alpha \in R$ we get a function $F\colon R \to R$ defined by $F(r) := \alpha r$, which is a homomorphism of abelian groups. In terms of the basis B, we will have formulas

$$F(e_j) = \sum_{i=1}^{m} c_{ij} e_i, \qquad c_{ij} \in \mathbb{Z}.$$

That is, F is represented in terms of the basis B by some matrix $C \in \mathrm{Mat}_{m\times m}(\mathbb{Z})$.

Now let $\alpha = a/b \in R \cap \mathbb{Q}$, where $a, b \in \mathbb{Z}$, $b \neq 0$. Then

$$a e_j = b\alpha e_j = bF(e_j) = b\sum_i c_{ij} e_j = \sum_i (b c_{ij} e_i).$$

Since B is a $\mathbb{Z}$-basis, it is $\mathbb{Z}$-linearly independent, so we can match coefficients of the e_is on both sides of the equation. In particular, equality of the coefficients of e_j gives

$$a = bc_{ij}, \qquad \text{so} \qquad \alpha = \frac{a}{b} = c_{ij} \in \mathbb{Z}$$

as desired. ☐

27. Induced characters

Let G be a group and H a subgroup, with $m = [G : H]$. Then we have a function

$$\mathrm{Res}_H^G \colon L^c(G) \to L^c(H)$$

which sends class functions on G to class functions on H, by restriction to the subgroup:

$$(\mathrm{Res}_H^G f)(h) := f(h), \qquad h \in H.$$

We call Res_H^G the **restriction** function. **restriction**

Proposition. *If χ is the character of a representation of G, then $\chi' = \mathrm{Res}_H^G \chi$ is the character of a representation of H.*

Proof. Let $\phi\colon G \to GL(V)$ be a reprentation with $\chi = \chi_\phi$. Let $\psi\colon H \to GL(V)$ the restriction of this map to the subgroup H. Then clearly the character of ψ is χ'. ☐

If $\chi_\psi = \mathrm{Res}_H^G \chi_\phi$, we say that ψ is the **restricted representation** of ϕ. **restricted representation**

It turns out there is a way to turn class functions on a subgroup intto class functions on the whole group. Given $H \leq G$ and $f \in L^c(H)$, we define $f' = \mathrm{Ind}_H^G f \in L^c(G)$ by the formula

$$f'(g) := \frac{1}{|H|} \sum_{\substack{x\in G \\ xgx^{-1}\in H}} f(xgx^{-1}).$$

This is a class function on G, since if $g' = tgt^{-1}$ for some $t \in G$, then

$$f'(tgt^{-1}) = \frac{1}{|H|} \sum_{\substack{x\in G \\ (xt)g(xt)^{-1}\in H}} f((xt)g(xt)^{-1}) = \frac{1}{|H|} \sum_{\substack{y\in G \\ ygy^{-1}\in H}} f(ygy^{-1}), \qquad y = xt.$$

Thus this construction defines a linear map

$$\mathrm{Ind}_H^G \colon L^c(H) \to L^c(G),$$

and is called the **induction** function. **induction**

Remark. Since f is a class function on H, if we replace x by xh, we get the same value in: $f(xgx^{-1}) = f(hxgx^{-1}h^{-1})$. It is thus often convenient to write the formula for induction as follows. Choose a set $R = \{x_i\}_{1\le i\le m}$ of representatives of right H-cosets in G (so $Hx = Hx_i$ for exactly one $x_i \in R$). Then the induced class function $f' = \operatorname{Ind}_H^G f$ is given by

$$f'(g) = \sum_{\substack{x_i\in R\\ x_i g x_i^{-1}\in H}} f(x_i g x_i^{-1}).$$

Exercise. Show that H is a normal subgroup of G iff for all $f \in L^c(H)$ the induced class function $f' = \operatorname{Ind}_H^G f \in L^c(G)$ vanishes on $G \smallsetminus H$.

Exercise. Let G be any finite group, and $H = \{e\} \le G$ the trivial subgroup. Compute $\chi' = \operatorname{Ind}_H^G \chi$, where χ is the trivial character on H. What representation of G does χ' correspond to?

Induction is "adjoint" to restriction, in the sense of linear algebra.

Proposition (Frobenius reciprocity)**.** *For any $f \in L^c(H)$ and $f' \in L^c(G)$, we have*

$$\langle \operatorname{Ind}_H^G f,\, f'\rangle_G = \langle f,\, \operatorname{Res}_H^G f'\rangle_H.$$

(I've put a subscript on the inner products to indicate that they are happening in different vector spaces, namely $L^c(G)$ and $L^c(H)$.)

Proof. Compute:

$$\begin{aligned}
\langle \operatorname{Ind}_H^G f,\, f'\rangle_G &= \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{\substack{x\in G\\ xgx^{-1}\in H}} f(xgx^{-1})\overline{f'(g)} \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{h\in H}\sum_{x\in G} f(h)\overline{f'(x^{-1}hx)} && \text{reindex sum by } h = xgx^{-1}, \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{h\in H}\sum_{x\in G} f(h)\overline{f'(h)} && f' \in L^c(G), \\
&= \frac{1}{|H|}\sum_{h\in H} f(h)\overline{f'(h)} = \langle f,\, \operatorname{Res}_H^G f'\rangle_H.
\end{aligned}$$

□

It turns out that induction also sends characters to characters.

Proposition. *If χ is the character of a representation of H, then $\chi' = \operatorname{Ind}_H^G \chi$ is the character of a representation of G.*

If $\chi_\psi = \operatorname{Ind}_H^G \chi_\phi$, we say that ψ is the **induced representation** of ϕ. Note that by the formula for induction, $\chi'(e) = [G : H]\chi(e)$ and thus $\dim\psi = [G : H]\dim\phi$. **induced representation**

We will prove the existence of induced representations soon.

Example. Let ψ be the irreducible 2-dimensional representation of S_3, with character

$$\chi(e) = 2, \qquad \chi((1\ 2)) = 0, \qquad \chi((1\ 2\ 3)) = -1.$$

We can regard S_3 as a subgroup of S_4 (i.e., as the subgroup of permutations of $\{1, 2, 3, 4\}$ which fix 4), and so we can form $\chi' = \operatorname{Ind}_{S_3}^{S_4} \chi$. We can compute this:

$$\chi'(e) = 8, \qquad \chi'((1\ 2)) = 0, \qquad \chi'((1\ 2)(3\ 4)) = 0, \qquad \chi'((1\ 2\ 3\ 4)) = 0, \qquad \chi'((1\ 2\ 3)) = -1.$$

Note that (1 2)(3 4) and (1 2 3 4) are not conjugate in S_4 to any elements of S_3, and (1 2) is only conjugate to 2-cycles, on which χ gives 0. For the remaining cases,

$$\chi'(e) = \frac{1}{6} \sum_{\substack{x \in G \\ xex^{-1} \in S_3}} \chi(xex^{-1}) = \frac{24}{6}\chi(e) = 8,$$

and

$$\chi'((1\ 2\ 3)) = \frac{1}{6} \sum_{\substack{x \in G \\ x(1\ 2\ 3)x^{-1} \in S_3}} \chi(x(1\ 2\ 3)x^{-1}) = \frac{1}{6}\sum_{x \in S_3} \chi(x(1\ 2\ 3)x^{-1}) = \chi(1\ 2\ 3) = -1.$$

So there is is an 8-dimensional representation ψ of S_4 with this character. It is not irreducible: $\langle \chi', \chi' \rangle_{S_4} = 3$. Since the only way to write this as a sum of squares is $3 = 1^2 + 1^2 + 1^2$, this will be the sum of three distinct irreducible S_4-representations. (*Exercise:* determine these irreducibles, by referring to the character table for S_4 given earlier.)

Exercise. Let $H \leq G$. Show that if $\chi \in L^c(H)$ is the trivial character ($\chi(h) = 1$ for all $h \in H$), then $\chi' = \mathrm{Ind}_H^G$ is the character of the permutation representation $\rho\colon G \to GL(\mathbb{C}(G/H))$ of the left-coset action by G on G/H.

28. Character table for D_5

Let's start with a character table for $\mathbb{Z}_5$, which we think of as the cyclic subgroup $\langle r \rangle \leq D_5$.

	e	r	r^2	r^3	r^4
χ_1	1	1	1	1	1
χ_2	1	ζ	ζ^2	ζ^3	ζ^4
χ_3	1	ζ^2	ζ^4	ζ	ζ^3
χ_4	1	ζ^3	ζ	ζ^4	ζ^2
χ_5	1	ζ^4	ζ^3	ζ^2	ζ

where $\zeta = e^{2\pi i/5}$.

We can induce these characters from $\langle r \rangle$ to D_5, the dihedral group of order 10.

10	1	2	2	5
	e	r	r^2	j
$\mathrm{Ind}\,\chi_1$	2	2	2	0
$\mathrm{Ind}\,\chi_2$	2	α	β	0
$\mathrm{Ind}\,\chi_3$	2	β	α	0
$\mathrm{Ind}\,\chi_4$	2	β	α	0
$\mathrm{Ind}\,\chi_5$	2	α	β	0

where $\alpha = \zeta + \zeta^{-1} = 2\cos 2\pi/5$ and $\beta = \zeta^2 + \zeta^{-2} = 2\cos 4\pi/5$. Note that

$$\alpha\overline{\alpha} + \beta\overline{\beta} = \alpha^2 + \beta^2 = (\zeta^2 + 2 + \zeta^{-2}) + (\zeta^{-1} + 2 + \zeta) = 3 + (1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4) = 3,$$

from which we see that $\langle \mathrm{Ind}\,\chi_k, \mathrm{Ind}\,\chi_k \rangle = 1$ for $k = 2, 3, 4, 5$. However, $\mathrm{Ind}\,\chi_1$ is not irreducible, since $\langle \mathrm{Ind}\,\chi_1, \mathrm{Ind}\,\chi_1 \rangle = 2$.

Using this, we can fill in the whole character table for D_5.

10	1	2	2	5
	e	r	r^2	j
χ'_1	1	1	1	1
χ'_2	1	1	1	-1
χ'_3	2	α	β	0
χ'_4	2	β	α	0
	10	5	5	2

29. Construction of induced representations

Let $H \leq G$. Given a representation $\phi\colon H \to GL(V)$ with character χ, we want to construct a representation $\psi\colon G \to GL(W)$ with character $\chi' = \operatorname{Ind}_H^G \chi$.

Let

$$W := \{\, \omega\colon G \to V \mid \omega(hx) = \phi_h(\omega(x)) \text{ for all } x \in G,\ h \in H \,\},$$

the set of all "H-equivariant functions" from the set G to the vector space V. The set W is a vector subspace of the set of all functions $G \to V$.

Let $R = \{x_i\} \subseteq G$ be a set of representatives of the right H-cosets in G (so $Hx = Hx_i$ for exactly one $x_i \in R$). We see that a function $\omega \in W$ is exactly determined by its values $\omega(x_i)$ for $x_i \in R$. In fact, this shows that $\dim W = [G : H] \dim V$.

Define $\psi\colon G \to GL(W)$ by

$$\psi_g(\omega)(x) := \omega(xg), \qquad g \in G, \quad \omega \in W, \quad x \in G.$$

Lemma. *ψ is a representation of G.*

Proof. The key part is to show that it is well-defined: that $\omega \in W$ implies $\omega' = \psi_g(\omega) \in W$. This is straightforward: if $h \in H$, then

$$\omega'(hx) = \omega(hxg) = \phi_h(\omega(xg)) = \phi_h(\omega'(x)).$$

Then we can show that that ϕ is a homomorphism, since $\psi_{g_1}(\psi_{g_2}(\omega))(x) = \psi_{g_2}(\omega)(xg_1) = \omega(xg_1g_2) = \psi_{g_1g_2}(\omega)(x)$. $\square$

Now we need to compute the character of ψ. For a right H-coset $Hx \subseteq G$, let

$$W_{Hx} = \{\, \omega \in W \mid \omega(y) = 0 \text{ if } y \notin Hx \,\},$$

the subspace of functions in W which are "supported" on the coset Hx.

Lemma. *The vector space W is a direct sum of the collection of subspaces W_{Hx_i}, where $x_i \in R$, i.e., $W = W_{Hx_1} \oplus \cdots \oplus W_{Hx_m}$, where $m = [G : H]$.*

Proof. We need to show that the map $\pi\colon W_{Hx_1} \oplus \cdots \oplus W_{Hx_m} \to W$ sending $(\omega_1, \dots, \omega_m) \mapsto \sum_i \omega_i$ is an isomorphism of vector spaces.

Given $\omega \in W$ and $x_i \in R$, let

$$\omega_i(g) := \begin{cases} \omega(g) & \text{if } g \in Hx_i, \\ 0 & \text{if } g \notin Hx_i. \end{cases}$$

This ω_i satisfies $\omega_i(hx) = \phi_h(\omega_i(g))$ for all $g \in G$ and $h \in H$, since $g \in Hx_i$ iff $hg \in Hx_i$. Therefore $\omega_i \in W_{Hx_i}$ by construction. Since $\omega = \sum_i \omega_i$, the map π is surjective.

If $\beta_i \in W_{Hx_i}$ are such that $\beta := \sum_i \beta_i = 0$, then for any $g \in Hx_i$ we have $\beta_i(g) = \beta(g) = 0$, whence $\beta = 0$. Thus π is injective. $\square$

Lemma. *We have that $\psi_g(W_{Hx}) \subseteq W_{Hxg^{-1}}$. In particular, $\psi_g(W_{Hx}) \subseteq W_{Hx}$ iff $xgx^{-1} \in H$.*

Proof. Suppose $\omega \in W_{Hx}$ and let $\omega' := \psi_g(\omega)$. Then for $y \notin Hxg^{-1}$ we have $\omega'(y) = \omega(yg) = 0$, since $yg \notin Hx$. $\square$

Thus, in terms of the direct sum decomposition $W = W_{Hx_1} \oplus \cdots \oplus W_{Hx_m}$, the operator ψ_g has a "block matrix" decomposition $(\psi_{ij}(g))$ where $\psi_{ij}(g) \in \operatorname{Hom}(W_{Hx_j}, W_{Hx_i})$, with the property that in each row and column only one "entry" $\psi_{ij}(g)$ is non-zero. To compute the trace, we only need to worry about the "diagonal" entries ψ_{ii}, and thus

$$\operatorname{Tr}(\psi_g) = \sum_{x_i \in R} \operatorname{Tr}(\psi_{ii}(g)) = \sum_{\substack{x_i \in R \\ x_i g x_i^{-1} \in H}} \operatorname{Tr}(\psi_g|_{W_{Hx_i}}),$$

since $\psi_{ii}(g)$ is non-zero only if $x_i g x_i^{-1} \in H$.

For any $x \in G$ we can define a linear map

$$E_x \colon W_{Hx} \to V, \qquad E_x\omega := \omega(x)$$

by evaluation at x.

Lemma. *The map $E_x \colon W_{Hx} \to V$ is an isomorphism of vector spaces.*

Proof. The inverse function is defined by

$$E_x^{-1}(v)(y) = \begin{cases} \phi_h(v) & \text{if } y = hx \text{ for some } h \in H, \\ 0 & \text{if } y \notin Hx. \end{cases}$$

□

Lemma. *If $g, x \in G$ are such that $xgx^{-1} \in H$, then $(\psi_g|_{W_{Hx}}) = E_x^{-1}\phi_{xgx^{-1}}E_x$.*

Proof. Let $\omega \in W_{Hx}$. Then

$$E_x(\psi_g(\omega)) = \psi_g(\omega)(x) = \omega(xg), \qquad \phi_{xgx^{-1}}(E_x(\omega)) = \phi_{xgx^{-1}}(\omega(x)) = \omega(xgx^{-1}x) = \omega(xg).$$

□

Remark. The same argument shows more generally that, when $ygx^{-1} \in H$, so that $\psi(W_{Hx}) \subseteq W_{Hy}$, we have

$$\psi_g|_{Hx} = E_y^{-1}\phi_{ygx^{-1}}E_x.$$

This gives a formula for all the non-zero "blocks" of ψ_g.

As a consequence, we get that

$$\mathrm{Tr}(\psi_g) = \sum_{\substack{x_i \in R \\ x_i g x_i^{-1} \in H}} \mathrm{Tr}(\phi_{x_i g x_i^{-1}}),$$

and therefore $\chi_\psi = \mathrm{Ind}_H^G \chi_\phi$ as desired.

30. Representations of products of groups

Let G_1 and G_2 be groups, and let $H = G_1 \times G_2$ be their product.

Given class functions $f_1 \in L^c(G_1)$ and $f_2 \in L^c(G_2)$, we can produce a new class function $f_1 \otimes f_2 \in L^c(H)$ on the product, by the formula

$$(f_1 \otimes f_2)(g_1, g_2) := f_1(g_1)f_2(g_2).$$

The new class function $f_1 \otimes f_2$ is called the **product** of the characters. It is often just written as "$f_1 f_2$". **product**

We have a formula for inner products of product characters.

Proposition. *If $f_1, f_1' \in L^c(G_1)$ and $f_2, f_2' \in L^c(G_2)$, then*

$$\langle f_1 \otimes f_2,\, f_1' \otimes f_2' \rangle_H = \langle f_1,\, f_1' \rangle_{G_1} \langle f_2,\, f_2' \rangle_{G_2}.$$

Proof. This is a straightforward calculation:

$$\begin{aligned}
\langle f_1 \otimes f_2,\, f_1' \otimes f_2' \rangle_H &= \frac{1}{|H|} \sum_{h \in H} (f_1 \otimes f_2)(h)\, \overline{(f_1' \otimes f_2')(h)} \\
&= \frac{1}{|G_1|\,|G_2|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} f_1(g_1) f_2(g_2) \overline{f_1'(g_1) f_2'(g_2)} \\
&= \left(\frac{1}{|G_1|} \sum_{g_1 \in G_1} f_1(g_1) \overline{f_1'(g_1)} \right) \left(\frac{1}{|G_2|} \sum_{g_2 \in G_2} f_2(g_2) \overline{f_2'(g_2)} \right) \\
&= \langle f_1,\, f_1' \rangle_{G_1} \langle f_2,\, f_2' \rangle_{G_2}.
\end{aligned}$$

□

The tensor product of characters is also a character.

Proposition. *If χ is the character of a representation of G_1, and χ' is the character of a representation of G_2, then $\chi \otimes \chi'$ is the character of a representation of $H = G_1 \otimes G_2$.*

If $\chi_\rho = \chi_\phi \otimes \chi_\psi$, we say that ρ is the **tensor product representation** of ϕ and ψ. Note that by the formula, $\dim \rho = (\dim \phi)(\dim \psi)$.

tensor product representation

I'll prove the existence of tensor product representations soon. First let's get the big consequence: if a group is a product, its representations are determined by the representations of its factors.

Corollary. *Let $H = G_1 \times G_2$. Let $\chi_1, \ldots, \chi_r$ be a complete set of pairwise distinct irreducible characters of G_1, and $\chi_1', \ldots, \chi_s'$ be a complete set of pairwise distinct irreducible characters of G_2. Then*

$$\{\, \chi_i \otimes \chi_j' \mid 1 \le i \le r,\ 1 \le j \le s \,\}$$

is a complete set of pairwise distinct irreducible characters of H.

Proof. First, note that $\{\chi_i \otimes \chi_j'\}$ is an orthornormal subset of $L^c(H)$:

$$\begin{aligned}
\langle \chi_i \otimes \chi_j',\, \chi_k \otimes \chi_\ell' \rangle_H &= \langle \chi_i,\, \chi_k' \rangle_{G_1} \langle \chi_j,\, \chi_\ell' \rangle_{G_2} \\
&= \delta_{ik} \delta_{j\ell}.
\end{aligned}$$

In particular, each $\chi_i \otimes \chi_j'$ is an irreducible character.

To see that it is a complete set, it suffices to show that $rs = \dim L^c(H) =$ number of conjugacy classes in H. If $h = (g_1, g_2) \in H$, then its conjugates in H are the elements of the form $(a_1, a_2)(g_1, g_2)(a_1, a_2)^{-1} = (a_1 g_1 a_1^{-1}, a_2 g_2 a_2^{-1})$, for $a_1 \in G_1$ and $a_2 \in G_2$. Thus

$$\mathrm{Cl}_H(h) = \{\, (x_1, x_2) \in H \mid x_1 \in \mathrm{Cl}_{G_1}(g_1),\ x_2 \in \mathrm{Cl}_{G_2}(g_2) \,\} = \mathrm{Cl}_{G_1}(g_1) \times \mathrm{Cl}_{G_2}(g_2).$$

Thus each conjugacy class C in H corresponds to exactly one pair (C_1, C_2), where C_k is a conjugacy class in G_k. □

Remark. If $H = G \times G$, then we can identify G with the **diagonal subgroup** Δ of the product, defined by $\Delta := \{\, (g, g) \mid g \in G \,\}$. By the above, if χ_1, χ_2 are characters on G, then

diagonal subgroup

$$\chi := \mathrm{Res}_{\Delta}^{G \times G} (\chi_1 \otimes \chi_2)$$

is also a character on G, with formula

$$\chi(g) = \chi_1(g) \chi_2(g).$$

We have already seen a special case of this, when one of the original characters is 1-dimensional.

As a consequence, the subset $X(G) \subseteq L^c(G)$ of characters has operations of addition and multiplication (defined in the usual way for functions) which are associative and commutative. Furthermore, multiplication distributes over addition, and these operations have identity elements (the characters of the 0-representation and the trivial representation respectively). Thus $X(G)$ has the structure of a **commutative semi-ring**. (It is not closed under additive inverses, so it is not a commutative ring.) **commutative semi-ring**

Although the subset of characters is not a ring, you can enlarge it to get a ring. A **virtual character** is any function which is a difference $\chi_\phi - \chi_\psi$ of two characters. You can show that the subset $R(G) \subseteq L^c(G)$ of class functions which are virtual characters is actually a subring. It is called the **representation ring** of G. **virtual character** **representation ring**

Now let's prove the proposition.

Construction of tensor product representations. Let $\phi\colon G_1 \to GL(V)$ and $\psi\colon G_2 \to GL(W)$ be representations. Choose bases $v_1, \dots, v_m$ of V and $w_1, \dots, w_n$ of W. Using these bases we can rewrite these as matrix representations, so that

$$\phi_x(v_j) = \sum_{i=1}^m \phi_{ij}(x) v_i, \qquad \psi_y(w_j) = \sum_{i=1}^n \psi_{ij}(y) w_i.$$

Let U be any vector space of dimension mn, and choose a basis $\{\, u_{i,k} \mid 1 \le i \le m,\ 1 \le k \le n \,\}$ of U. Define $\rho\colon G_1 \times G_2 \to GL(U)$ by

$$\rho_{(x,y)}(u_{j,\ell}) := \sum_{i=1}^m \sum_{k=1}^n \phi_{ij}(x)\psi_{k\ell}(y) u_{i,k}.$$

It is straightforward (but a little tedious), to show that this is a representation:

$$\begin{aligned}
\rho_{(x,y)}(\rho_{(x',y')}(u_{j,\ell})) &= \rho_{(x,y)}\left(\sum_{i=1}^m \sum_{k=1}^n \phi_{ij}(x')\psi_{k\ell}(y') u_{i,k}\right) \\
&= \sum_{i=1}^m \sum_{k=1}^n \phi_{ij}(x')\psi_{k\ell}(y') \rho_{(x,y)}(u_{i,k}) \\
&= \sum_{i=1}^m \sum_{k=1}^n \phi_{ij}(x')\psi_{k\ell}(y') \sum_{s=1}^m \sum_{t=1}^n \phi_{si}(x)\psi_{tk}(y) u_{s,t} \\
&= \sum_{s=1}^m \sum_{t=1}^n \left(\sum_{i=1}^m \phi_{si}(x)\phi_{ij}(x')\right)\left(\sum_{k=1}^n \psi_{tk}(y)\psi_{k\ell}(y')\right) u_{s,t} \\
&= \sum_{s=1}^m \sum_{t=1}^n \phi_{sj}(xx')\psi_{t\ell}(yy') u_{s,t} \\
&= \rho_{(xx',yy')}(u_{j,\ell}).
\end{aligned}$$

To compute the character of ρ, note that the coefficient of $u_{j,\ell}$ in $\rho_{(x,y)}(u_{j,\ell})$ is $\phi_{jj}(x)\psi_{\ell\ell}(y)$, so

$$\begin{aligned}
\operatorname{Tr}(\rho_{(x,y)}) &= \sum_{j=1}^m \sum_{\ell=1}^n \phi_{jj}(x)\psi_{\ell\ell}(y) \\
&= \left(\sum_{j=1}^m \phi_{jj}(x)\right)\left(\sum_{\ell=1}^n \psi_{\ell}(y)\right) \\
&= \operatorname{Tr}(\phi_x) \operatorname{Tr}(\psi_y),
\end{aligned}$$

so $\chi_\rho = \chi_\phi \otimes \chi_\psi$. ☐

Example (Character table of D_6). Note that D_6 is a product group of the subgroups $\langle r^2, j\rangle \approx S_3$ and $\langle r^3\rangle \approx \mathbb{Z}_2$.

6	1	3	2
$\langle r^2, j\rangle$	e	j	r^2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
	6	2	3

2	1	1
$\langle r^3\rangle$	e	r^3
χ_1'	1	1
χ_2'	1	-1
	2	2

Thus we get the following table for the product.

12	1	3	2	1	3	2
D_6	e	j	r^2	r^3	jr^3	r^5
$\langle r^2, j\rangle \times \langle r^3\rangle$	(e,e)	(j,e)	(r^2,e)	(e,r^3)	(j,r^3)	(r^2,r^3)
$\chi_1 \otimes \chi_1'$	1	1	1	1	1	1
$\chi_2 \otimes \chi_1'$	1	-1	1	1	-1	1
$\chi_3 \otimes \chi_1'$	2	0	-1	2	0	-1
$\chi_1 \otimes \chi_2'$	1	1	1	-1	-1	-1
$\chi_2 \otimes \chi_2'$	1	-1	1	-1	1	-1
$\chi_3 \otimes \chi_3'$	2	0	-1	-2	0	1
	12	4	6	12	4	6

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