MATH 427 09/28/2022

16.1

Last time: • 2nd iso thm: G a group, $H < G$, $N \lhd G$. Then $HN = \{hn \mid h \in H, n \in N\}$ is a subgroup of G and $f : H/H\cap N \to HN/N$, $f(a(H\cap N)) = aN$ is a well-defined isomorphism.

- 3d isomorphism thm: G a group, $K, H \lhd G$, $K \subseteq H$. Then $H/K \lhd G/K$ and $f : (G/K)/H/K \to G/H$
$$f((gK)\,H/K) = gH$$
is a well-defined isomorphism.

Goal: There is a unique homomorphism $\text{sign} : S_n \to \{\pm 1\}$ s.t. $\text{sign}((ij)) = -1$ for any transposition (ij).
(Recall: a transposition is a 2-cycle)

Lemma 16.1 Every permutation is a product of transpositions. That is, $S_n = \langle (ij) \mid 1 \le i < j \le n \rangle$.

Proof Any $\sigma \in S_n$ is a product of cycles. So enough to prove:

$$(*) \quad (i_1 \dots i_r) = (i_1 i_2) \dots (i_{r-2}, i_{r-1})(i_{r-1}\ i_r) \quad \forall\, r > 1 \ \forall\, i_1, \dots i_r \in \{1, \dots n\}$$

Proof of (*). If $j \in \{1, \dots n\}$ and $j \neq i_1, \dots i_r$ then $(RHS)(j) = LHS(j)$.
If $j = i_r$, then $(LHS)(j) = i_1$
$RHS(j) = \quad i_r \mapsto i_{r-1} \mapsto i_{r-2} \mapsto \dots \mapsto i_1$
Similarly $(RHS)(i_s) = (LHS)(i_s)$ for $1 \le s < i_r$, $\Rightarrow$ (*) holds $\square$

Lemma 16.2 sign w. desired properties is unique

Proof Suppose $f : S_n \to \{\pm 1\}$ is another homomorphism with $f((ij)) = -1 \quad \forall\, (ij) \in S_n$.

Let $\sigma \in S_n$. Then $\exists\, k$, transpositions $\tau_1, \dots \tau_k$ s.t. $\sigma = \tau_1 \dots \tau_k$
Note: no claim that τ's are unique!

Then $f(\sigma) = f(\tau_1) \dots f(\tau_k) = (-1)^k = \text{sign}(\tau_1) \dots \text{sign}(\tau_k)$
$= \text{sign}(\sigma)$

Remark There are two common proofs of existence of sign. One proof is : if $\sigma = \tau_1 \cdots \tau_k = \mu_1 \cdots \mu_n$ where τ_i's, μ_i's are transpositions then $k \equiv n \bmod 2$.

We'll proceed differently.

Lemma 16.3 There is a homomorphism $\rho : S_n \to GL(n, \mathbb{R})$ with

$$\rho(\,(ij)) = \begin{array}{cc} \left(\begin{array}{ccccccc} 1 &&&&&& \\ & \ddots &&&&& \\ && 0 && 1 && \\ &&& \ddots &&& \\ && 1 && 0 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{array}\right) & \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} && i && j && \end{array} & \end{array} \qquad \forall\ 1 \le i < j \le n.$$

Proof Any linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is uniquely determined by its values on the standard basis $\{e_1, \dots, e_n\}$.

Define $\rho(\sigma) : \mathbb{R}^n \to \mathbb{R}^n$ by $\rho(\sigma)\, e_i = e_{\sigma(i)} \quad \forall i$

Then for $\sigma, \mu \in S_n$

$$\rho(\sigma\mu)\, e_i = e_{(\sigma\mu)(i)} = e_{\sigma(\mu(i))} = \rho(\sigma)\,(e_{\mu(i)}) =$$

$$= \rho(\sigma)\,(\rho(\mu)\,(e_i)) = (\rho(\sigma)\,\rho(\mu))\,(e_i)$$

$$\Rightarrow \quad \rho(\sigma\mu) = \rho(\sigma) \circ \rho(\mu).$$

For $\sigma = (ij)$, $\rho(\sigma)\, e_k = \begin{cases} e_k & k \neq i,j \\ e_i & k = j \\ e_j & k = i \end{cases} \quad \to \rho(\sigma) = \begin{array}{cc} \begin{array}{ccccccc} && i && j && \end{array} & \\ \left(\begin{array}{ccccccc} 1 &&&&&& \\ & \ddots &&&&& \\ && 0 && 1 && \\ &&& \ddots &&& \\ && 1 && 0 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{array}\right) & \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \end{array}$

Remark $\forall\, \sigma \in S_n \quad \forall\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$\rho(\sigma)x = \rho(\sigma)\,(\Sigma\, x_i e_i) = \sum_i x_i e_{\sigma(i)} = \left(\begin{array}{c} j = \sigma(i) \\ i = \sigma^{-1}(j) \end{array} \quad \text{"change of variable"} \right)$$

$$= \sum_i x_{\sigma^{-1}(i)}\, e_i$$

Thus $\rho(\sigma)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}$

Theorem 16.4 There is a homomorphism $\text{sign} : S_n \to \{\pm 1\}$ so that $\text{sign}(\,(ij)) = -1 \ \forall\, i,j$. (By 16.2 sign is unique)

<u>Proof?</u> Define $\text{sign}(\sigma) = \det(\rho(\sigma))$ where ρ is the permutation representation. <u>Problem: det</u> is often (not always!) is defined by $\det((a_{ij})) = \sum_{\sigma \in S_n} \text{sign}\,\sigma \; a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)}$.
This is circular.

<u>Proof</u> Consider $P : \mathbb{R}^n \to \mathbb{R}$, $P(x_1, \ldots, x_n) := \prod_{i<j} (x_i - x_j)$

(For example $P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$)

Define

$$\text{sign}(\sigma) = \frac{P\left(\rho(\sigma)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)}{P(x_1 \ldots x_n)} = \frac{\prod_{i<j} (x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(j)})}{\prod_{i<j} (x_i - x_j)}$$

<u>Ex</u> $n = 3$ $\sigma = (123)$ $\sigma^{-1} = (321)$ $\rho(\sigma)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$

$$\text{sign}((123)) = \frac{P(x_3, x_1, x_2)}{P(x_1, x_2, x_3)} = \frac{(x_3 - x_1)(x_3 - x_2)\cancel{(x_1 - x_3)}}{(x_1 - x_2)\cancel{(x_1 - x_3)}(x_2 - x_3)} = (-1)^2 = 1$$

$\forall\, \sigma, \mu \in S_n$

$$\text{sign}(\sigma\mu) = \frac{P(\rho(\sigma\mu)x)}{P(x)} = \frac{P(\rho(\sigma)\overbrace{(\rho(\mu)x)}^{y})}{P(\underbrace{\rho(\mu)x}_{y})} \cdot \frac{P(\rho(\mu)x)}{P(x)}$$

$$= \frac{P(\rho(\sigma)y)}{P(y)} \frac{P(\rho(\mu)x)}{P(x)} = \text{sign}(\sigma) \cdot \text{sign}(x).$$

$\therefore$ $\text{sign} : S_n \to \{\pm 1\}$ is a homomorphism.

Remains to show: $\text{sign}((kl)) = -1$ $\forall\, (kl) \in S_n$.

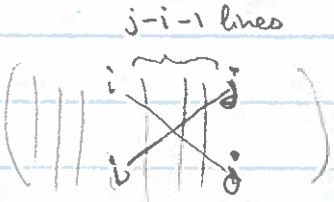
<u>Definition</u> Let $\sigma \in S_n$ be a permutation. An <u>inversion</u> in σ is a pair $i, j \in \{1, \ldots, n\}$ s.t. $i < j$ and $\sigma(i) > \sigma(j)$

The <u>inversion number of σ</u> is the # of inversions of σ.

<u>Ex</u> $\sigma = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ inversion # = # of crossings
in this case inversion # = 2.

<u>Observation</u> Inversion # of a transposition (ij), $i<j$ is $(j-i-1)\cdot 2+1$.

Proof by picture:



Finally $$\frac{P(\rho(\sigma)x)}{P(x)} = (-1)^{|\{(i,j) \mid i<j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}|}$$
$$= (-1)^{\text{inversion \# of } \sigma^{-1}}$$

Hence $$\operatorname{sign}((k\ell)) = (-1)^{\text{inversion \# of } (k\ell)^{-1}} = (-1)^{\text{inversion \# of } (k\ell)}$$
$$= (-1)^{(k-\ell-1)\cdot 2+1} = -1$$ □

<u>Defn</u> $A_n := \ker(\operatorname{sign} : S_n \to \{\pm 1\})$, <u>the alternating group on n letters.</u> $(n>1)$

Remark $A_n \lhd S_n$ and $S_n/A_n \cong \{\pm 1\}$

Hence $|A_n|\cdot 2 = |S_n| = n! \Rightarrow |A_n| = \frac{1}{2}n!$