MATH 427

07/26/2022
Mon

15.1

<u>Last time</u> (1) Let N be a normal subgroup of a group G. Then

$$*: G/N \times G/N \to G/N, \quad (aN)*(bN) = (ab)N$$

is a well-defined binary operation, and $(G/N, *, N=eN)$ is a group

(2) [1st isomorphism thm] Let $f: G \to H$ be a homomorphism. Then

$$\bar{f}: G/\ker f \to \operatorname{im} f, \quad \bar{f}(a \ker f) = f(a)$$

is a well-defined isomorphism. Moreover

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\ker f \arrow[r, "\bar{f}"] & \operatorname{im} f \arrow[u, "i"'] \end{tikzcd}$$

commutes, where $\pi(g) = g\ker f$ and $i(x) = x \ \forall x \in \operatorname{im} f$.

<u>Remark 1</u> Suppose $N \trianglelefteq G$, $a, b \in G$. One <u>can</u> (but shouldn't) define $*: G/N \times G/N \to G/N$ by $(aN)*(bN) = \{xy \mid x \in aN, y \in bN\}$

It's equivalent to: $(aN)*(bN) = (ab)N$.

However, with the alternative definition it's much harder to prove that 1) $*$ is associative (2) N is the identity and (3) G/N is a group and that (4) $\pi: G \to G/N$, $\pi(a) = aN$ is a homomorphism.

<u>Remark 2</u> If $N \trianglelefteq G$ then $\pi: G \to G/N$, $\pi(a) = aN$ is surjective. Then $\ker \pi = N$ and 1st iso theorem says

$$\bar{\pi}: G/N \to G/N, \quad \bar{\pi}(aN) = aN \text{ is an iso.}$$

<u>Third isomorphism Theorem</u> Let $K \subseteq H$ be two normal subgroups of a group G. Then $H/K \trianglelefteq G/K$ and

$$(G/K)/(H/K) \text{ is isomorphic to } G/H.$$

<u>To</u> prove 3rd iso theorem we first prove:

<u>Lemma 15.1</u> Suppose $f: G \to L$ is a homomorphism, $N \trianglelefteq G$ and $N \subseteq \ker f$. Then $\tilde{f}: G/N \to L'$, $\tilde{f}(aN) = f(a)$ is a well-defined homomorphism.

<u>Proof</u> Suppose $a_1 N = a_2 N$. Then $a_1 = a_2 n$ for some $n \in N$
$\Rightarrow f(a_1) = f(a_2 n) = f(a_2) f(n) = f(a_2)$ since $n \in N \subseteq \ker f$.
$\therefore$ $\tilde{f}$ is well-defined.

<u>Proof</u> (of 3^d iso thm) Consider $f: G \to G/H$, $f(g) = gH$. Then $H = \ker f$. Since $K \subseteq H$, $\tilde{f}: G/K \to G/H$, $\tilde{f}(aK) = aH$ is a well-defined homomorphism by 15.1. Since f is onto, so is $\tilde{f}$. $\ker \tilde{f} = \{aK \mid aH = H\} = \{aK \mid a \in H\} = H/K$.
Hence H/K is a normal subgroup of G/K and, by 1st iso theorem

$$\bar{f}: (G/K)/(H/K) \to G/H$$
$$\bar{f}((aK)(H/K)) = aH$$

is an isomorphism.

<u>Ex</u> $G = \mathbb{Z}$, $H = 3\mathbb{Z}$, $K = 6\mathbb{Z}$, 3^d iso theorem $\Rightarrow$

$$\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} \cong (\mathbb{Z}/6\mathbb{Z})/(3\mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}_6/\{[0],[3]\}$$
$$= \mathbb{Z}_6/\langle [3] \rangle.$$

<u>2nd isomorphism theorem</u> Let G be a group, $H < G$ a subgroup, $N \trianglelefteq G$ a normal subgroup. Then

$$\langle H \cup N \rangle = \{hn \mid h \in H, n \in N\} =: HN$$

Furthermore $H \cap N \trianglelefteq N$ and

$$H/H \cap N \xrightarrow{\cong} HN/N \quad \text{(isomorphic groups)}$$

(1) If K is any subgroup of G with $H \cup N \subseteq K$, Then $\forall h \in H, \forall n \in N$ $hn \in K$. Hence $HN \subseteq \langle H \cup N \rangle$.

If $x, y \in HN$ then $x = h_1 n_1$, $y = h_2 n_2$ for some $h_1, h_2 \in H$, some $n_1, n_2 \in N$.

Hence $xy^{-1} = (h_1 n_1)(h_2 n_2)^{-1} = h_1 n_1 n_2^{-1} h_2^{-1} = h_1 h_2^{-1} h_2 (n_1 n_2^{-1}) h_2^{-1} \in HN$

since $h_1 h_2^{-1} \in H$, $n_1 n_2^{-1} \in N$ and $gng^{-1} \in N$ $\forall g \in G, \forall n \in N$.

$\Rightarrow$ HN is actually a subgroup of G.

Now $\forall h \in H$, $he \in HN$. $\forall n \in N \Rightarrow en \in HN$. $\Rightarrow H, N \subseteq HN$.

$\therefore$ HN is the smallest subgroup of G containing H and N,

i.e. $HN = \langle H \cup N \rangle$.

(2) Consider the maps $i : H \to HN$, $i(h) = he$, $\pi : HN \to HN/N$

$\pi(x) = xN$. The two maps are homomorphisms. $\Rightarrow$

$$f = \pi \circ i : H \to HN/N, \quad f(h) = hN$$

is a homomorphism.

$\forall x \in HN$, $\exists h \in H, n \in N$ s.t. $x = hn$. Then $xN = hnN =$

$= (hN)(nN) = (hN) \cdot N = hN$.

$\Rightarrow$ $f : H \to HN/N$ is onto.

$\ker f = \{ h \in H \mid hN = N \} = \{ h \in H \mid h \in N \} = H \cap N$.

$\Rightarrow$ $\bar{f} : H/H \cap N \to HN/N$

$\bar{f}(h(H \cap N)) = hN$

is an isomorphism of groups by 1^{st} iso Theorem. $\square$

<u>Example</u> Let $G = GL(2, \mathbb{C}) = \{ A \in M_2(\mathbb{C}) \mid \det A \neq 0 \}$

Let $N = \ker(\det : GL(2,\mathbb{C}) \to \mathbb{C}^\times) =: SL(2, \mathbb{C})$.

Let $K = \mathbb{C}^\times \mathrm{id} = \{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \mid z \neq 0 \} = \{ z\,\mathrm{id} \mid z \in \mathbb{C}^\times \}$

<u>Note:</u> $\forall A \in GL(2, \mathbb{C})$ $A \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} A^{-1} = z A \cdot \mathrm{id} A^{-1} = z\,\mathrm{id}$

So $K \trianglelefteq GL(2, \mathbb{C})$

Also $\forall A \in GL(2,\mathbb{C})$ $\exists w \in \mathbb{C}^\times$ s.t. $w^2 = \det A$.

Then $A = (w\,\mathrm{id}) \cdot (\frac{1}{w} \mathrm{id}\, A)$ and $\det(\frac{1}{w} \mathrm{id}\, A) = \frac{1}{w^2} \det A = 1$

$$\Rightarrow \quad \tfrac{1}{w} A \in SL(2,\mathbb{C}). \quad \Rightarrow \quad GL(2,\mathbb{C}) = K \cdot SL(2,\mathbb{C}),$$

3^d iso Theorem $\to$

$$GL(2,\mathbb{C})/K = K \cdot SL(2,\mathbb{C})/K \simeq SL(2,\mathbb{C}) / K \cap SL(2,\mathbb{C})$$

$PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\mathbb{C}^\times id$ is called the group of projective linear transformations,

$$\mathbb{C}^\times id \cap SL(2,\mathbb{C}) = \{ z\, id \mid z^2 = 1\} = \{\pm 1\}\, id$$

Thus

$$PGL(2,\mathbb{C}) \simeq SL(2,\mathbb{C}) / \{\pm id\}.$$

$\square$

<u>Definition</u> A representation of a group G on a (real) vector space V is a homomorphism

$$\rho : G \to GL(V)$$

where $GL(V) = \{ T : V \to V \mid T \text{ linear isomorphism}\}$

We "represent" an element g of G by the linear transformation $\rho(g)$.