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11.1

Last time: $S_n := \mathrm{Sym}(\{1, \ldots n\})$, the group of bijections of the set $X = \{1, \ldots n\}$. The group operation is composition.

We've seen $|S_n| = n!$

Defined what it means for $f \in \mathrm{Sym}(X)$ to be a cycle.

Defined what it means for two cycles to be disjoint

We want to prove that any $\sigma \in S_n$ can be written uniquely as a product of disjoint cycles.

We need a bit of theory first.

Definition Let $G \times X \to X$ be a (left) action. The stabilizer of $x_0 \in X$ is
$$G_x \equiv \mathrm{Stab}(x) := \{g \in G \mid g \cdot x_0 = x_0\}.$$

Examples $GL(2,\mathbb{R})$ acts on $\mathbb{R}^2$.
$$\mathrm{Stab}(\vec{0}) = \{A \in GL(2,\mathbb{R}) \mid A\vec{0} = \vec{0}\} = GL(2,\mathbb{R})$$
$$\mathrm{Stab}\begin{pmatrix}1\\0\end{pmatrix} = \{A \in GL(2,\mathbb{R}) \mid A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\} = \{\begin{pmatrix}1 & a\\0 & b\end{pmatrix} \mid a,b \in \mathbb{R}, b \neq 0\}$$

Ex $\mathbb{R}$ acts on $\mathbb{C}$ by $\theta \cdot z = e^{2\pi i\theta} z$.
$$\mathrm{Stab}(0) = \{\theta \mid e^{2\pi i\theta} \cdot 0 = 0\} = \mathbb{R}.$$
Suppose $z \neq 0$. Then $e^{2\pi i\theta} z = z \Leftrightarrow e^{2\pi i\theta} = 1$
$$\Leftrightarrow \theta \in \mathbb{Z}$$
$$\Rightarrow \mathrm{Stab}(z) = \mathbb{Z}.$$

Lemma 11.1 Let $G \times X \to X$ be an action. For any $x \in X$ the stabilizer G_x is a subgroup of G.

Proof (i) By definition of the action, $e \cdot x = x \ \forall x \Rightarrow e \in G_x$.

(ii) Suppose $a, b \in G_x$, i.e. $a \cdot x = x$ and $b \cdot x = x$.
Then $(ab) \cdot x = a \cdot (b \cdot x) = a \cdot x = x. \Rightarrow ab \in G_x$.

(iii) Suppose $a \cdot x = x$. Then $a^{-1} \cdot x = a^{-1} \cdot (a \cdot x) = (a^{-1}a) \cdot x = e \cdot x = x$.

$\Rightarrow a^{-1} \in G_x$

$\therefore$ G_x is a subgroup of G.

<u>Theorem</u> (Orbit/stabilizer). Let $G \times X \to X$ be an action, $x \in X$

Then $f: G/G_x \to G \cdot x$, $f(gG_x) = g \cdot x$

is a well-defined bijection.

<u>Proof</u> (1) Suppose $gG_x = hG_x$. Then $g = ha$ for some $a \in G_x$.

$\Rightarrow$ $g \cdot x = (ha) \cdot x = h \cdot (a \cdot x) = h \cdot x$ (since $a \in G_x$)

$\therefore$ f is well-defined.

(2) Given $y \in G \cdot x$, $\exists g \in G$ st $y = g \cdot x$

$\Rightarrow$ $y = f(gG_x)$.

$\therefore$ f is onto.

(3) Suppose $f(gG_x) = f(hG_x)$.

Then $g \cdot x = h \cdot x$. $\Rightarrow$ $h^{-1} \cdot g \cdot x = h^{-1} \cdot h \cdot x = (h^{-1}h) \cdot x = x$

$\Rightarrow (h^{-1}g) \cdot x = x$

$\Rightarrow$ $h^{-1}g \in G_x$ $\Rightarrow$ $h^{-1}g = a$ for some $a \in G_x$

$\Rightarrow$ $g = ha$ for some $a \in G_x$ $\Rightarrow$ $g \in hG_x$ $\Rightarrow$ $gG_x = hG_x$.

$\therefore$ f is 1-1.

<u>Example</u> $\mathbb{R}$ acts on $\mathbb{C}$ by $\theta \cdot z = e^{2\pi i \theta} z$.

$\mathrm{Stab}(1) = \{\theta \mid e^{2\pi i \theta} \cdot 1 = 1\} = \mathbb{Z}$.

$\mathbb{R} \cdot 1 = \{e^{2\pi i \theta} \mid \theta \in \mathbb{R}\} = S^1$

Orbit/stabilizer theorem $\Rightarrow$ $\mathbb{R}/\mathbb{Z} \to S^1$, $\theta + \mathbb{Z} \mapsto e^{2\pi i \theta}$

is a well-defined bijection.

<u>Definition</u> A group G is <u>cyclic</u> if it is generated by one element:

$\exists g \in G$ st $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$.

We've [illegible]

We've proven: there is a surjective homomorphism

$$f : \mathbb{Z} \to \langle g \rangle, \quad f(n) = g^n.$$

If $|\langle g \rangle| = n$ then $\bar{f} : \mathbb{Z}_n \to \langle g \rangle$ $\bar{f}([k]) = g^k$ is a well-defined isomorphism.

<u>Back to permutations</u>

<u>Exercise</u> Suppose $f : X \to Y$ is a bijection between two sets. Consider $\varphi : Sym(X) \to Sym(Y)$, $\varphi(\tau) = f \circ \tau \circ f^{-1}$

$$\begin{tikzcd} X \arrow[r, "\tau"] & X \arrow[d, "f"] \\ Y \arrow[u, "f^{-1}"] \arrow[r, dashed, "\varphi(\tau)"'] & Y \end{tikzcd}$$

, Then φ is an isomorphism of groups.

<u>Hint</u> $\varphi^{-1}(\mu) = f^{-1} \circ \mu \circ f$ $\forall \mu : Y \to Y \in Sym(Y)$.

<u>Consequence</u> Let X be a set with n elements: $|X| = n$. Then there is a bijection $f : \{1, \dots, n\} \to X$. By exercise $\varphi : S_n \to Sym(X)$, $\varphi(\sigma) = f \circ \sigma \circ f^{-1}$ is an isomorphism.

We now prove that if X is finite then any $\tau \in Sym(X)$ is a product of disjoint cycles

<u>Recall:</u>

<u>Def.</u> Two bijections $\sigma, \tau \in Sym(X)$ are <u>disjoint</u> iff

$$\{x \in X \mid \sigma(x) \neq x\} \cap \{y \in X \mid \tau(y) \neq y\} = \emptyset.$$

<u>Lemma 11.2</u> Suppose $\sigma, \tau \in Sym(X)$ are disjoint. Then

$$\sigma \circ \tau = \tau \circ \sigma.$$

<u>Proof</u> We want to show: $\forall x \in X$, $\sigma(\tau(x)) = \tau(\sigma(x))$.

1. Suppose first $\sigma(x) \neq x$. Then
(i) $\tau(x) = x$ and (ii) $\sigma(\sigma(x)) \neq \sigma(x)$ since σ is 1-1.
Since σ and τ are disjoint, $\tau(\sigma(x)) = \sigma(x)$.
On the other hand $\tau(x) = x \Rightarrow \sigma(\tau(x)) = \sigma(x) = \tau(\sigma(x))$.

2. Suppose $\sigma(x) = x$. Then either $\tau(x) = x$ or $\tau(x) \neq x$.
If $\tau(x) = x$, $\tau(\sigma(x)) = \tau(x) = x = \sigma(x) = \sigma(\tau(x))$
If $\tau(x) \neq x$, then (1) applies with the roles of σ and τ switched.
$\Rightarrow \tau(\sigma(x)) = \sigma(\tau(x))$. $\square$

Recall A bijection $\sigma \in Sym(X)$ is a cycle of length $r > 1$ if
$\exists\, x_1, x_2, \ldots x_r \in X$, all distinct, s.t
$\sigma(x_1) = x_2$, $\sigma(x_2) = x_3 \ldots$ $\sigma(x_{r-1}) = x_r$ and $\sigma(x_r) = x_1$.

We want to prove: For any finite set X, any bijection $\sigma \in Sym(X)$ (with $\sigma \neq id$) can be written uniquely as a product of disjoint cycles.

Idea of proof Sym(X) acts on X. $\Rightarrow \langle \sigma \rangle \leq Sym(X)$ acts on X as well. Explicitly:

$$\sigma^k \cdot x = \begin{cases} \underbrace{\sigma \circ \cdots \circ \sigma}_{k}(x) & k > 0 \\ x & k = 0 \\ \underbrace{\sigma^{-1} \circ \cdots \circ \sigma^{-1}}_{|k|} & k < 0. \end{cases}$$

Orbits of the action of $\langle \sigma \rangle$ partition X.
Since X is finite there are only finitely many distinct orbits
$$X = \langle \sigma \rangle \cdot x^{(1)} \cup \ldots \cup \langle \sigma \rangle \cdot x^{(k)}$$
for some $x^{(1)}, \ldots x^{(k)} \in X$ with $\langle \sigma \rangle \cdot x^{(i)} \cap \langle \sigma \rangle \cdot x^{(j)} = \emptyset$ for $i \neq j$.

Moreover each orbit $\langle \sigma \rangle \cdot x^{(i)}$ is a finite and $\sigma \big|_{\langle \sigma \rangle \cdot x^{(i)}}$ is a cycle.