Last time: $O(n)$, $O(2) = SO(2) \cup \tau SO(2)$ where $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_n = subgroup of $O(2)$ generated by τ and $\rho = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$.

We can think of D_n as an "abstract" group with

$$D_n = \{ e, \rho, \dots \rho^{n-1}, \tau, \tau\rho, \dots \tau\rho^{n-1} \}$$

and multiplication rules: $\tau \cdot \tau = e$, $\rho^n = e$, $\tau\rho\tau^{-1} = \rho^{-1}$.

Or we can think of D_n concretely as acting on $\mathbb{C} \simeq \mathbb{R}^2$ with $\tau \cdot z = \bar{z}$ and $\rho \cdot z = e^{2\pi i/n} z$ $\quad \forall z \in \mathbb{C}$.

(By HW #3 problem 8 the "concrete" way of thinking of D_n identifies it with a subgroup of $\mathrm{Sym}(\mathbb{C}) = \mathrm{Sym}(\mathbb{R}^2)$.

Remark: $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication: $\lambda \cdot z = \lambda z$

So again by problem 8 we get a homomorphism

$$\varphi : \mathbb{C}^\times \to \mathrm{Sym}(\mathbb{C})$$

$\forall \lambda \in \mathbb{C}^\times \quad \varphi(\lambda) : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear.

$\Rightarrow$ We can think of $GL(2,\mathbb{R})$ as a subgroup of $\mathrm{Sym}(\mathbb{C})$.

and of φ as a homomorphism $\varphi : \mathbb{C}^\times \to GL(2,\mathbb{R})$

Note that $\varphi(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ because

i) $\{1, i\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$.

ii) $(a+ib)\cdot 1 = a+ib = a \cdot 1 + b \cdot i$

and $(a+ib)\cdot i = -b + ia = (-b)\cdot 1 + a \cdot i$

And then

$$\varphi(\rho) = \varphi(e^{2\pi i/n}) = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ +\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}.$$

Final comment about D_n: $\{1, e^{2\pi i/n}, e^{\frac{4\pi i}{n}}, \dots e^{\frac{2\pi i}{n}(n-1)}\}$ is the set of vertices of a regular n-gone.

Since $\forall a \in D_n \quad a \cdot \text{vertex} = \text{another vertex}$

The action of D_n on the n-gone maps it into itself. Thus D_n is a group of symmetries of a regular n-gone.

The permutation group(s) S_n, $n = 2, 3 \ldots$

Recall For any set X we have the group

$$Sym(X) = \{ f : X \to X \mid f \text{ invertible map} \}$$

The group operation is composition: $\forall f, g \in Sym(X)$ $f \cdot g = f \circ g$

The identity element is id_X; $id_X(x) = x \ \forall x \in X$.

Special case: $n \geq 2$, $X = \{1, 2, \ldots n\}$. $S_n := Sym(\{1, \ldots n\})$ the symmetric group on n letters

Elements of S_n are called permutations

Lemma 10.1 S_n has $n!$ elements: $|S_n| = n!$

(For a set X, $|X|$ = # of elements of X, cardinality of X)

Proof

Given $\sigma \in S_n$ there are n choices for $\sigma(1)$

$n-1$ choices for $\sigma(2)$ (σ is a bijection!)

$\vdots$

1 choice for $\sigma(n)$

Total # of choices is $n \cdot (n-1) \ldots 1 = n!$ $\square$

Notation We can picture $\sigma \in S_n$ as a table/array

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & & \sigma(n) \end{pmatrix}$$ these are not matrices!

For example

$$S_2 = \left\{ id_{\{1,2\}} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

There is a better notation, that uses <u>disjoint cycles</u>

<u>Ex</u> $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 3 & 2 & 7 & 1 & 5 \end{pmatrix} \in S_7$. What does σ do?

$1 \to 4 \to 2 \to 6 \quad 3 \quad 5 \to 7$

We write $\sigma = (1426)(3)(57) \overset{?}{=} (1426)(57)$ (drop (3))

<u>Definition</u> Let X be a set. $f \in Sym(X)$ is a <u>cycle of length r</u> iff $\exists\, x_1, \ldots x_r \in X$, all distinct so that

- $f(x_i) = x_{i+1}$ for $1 \le i < r$
- $f(x_r) = x_1$
- $f(x) = x$ for $x \neq x_1, \ldots x_r$.

$\boxed{\text{If } f \text{ is such a cycle we write } f = (x_1, \ldots x_r).}$

<u>Ex</u> $S_2 = \{ id, (12) \}$. $S_3 = \{ id, (12), (13), (23), (123), (132) \}$.

<u>Definition</u> Two cycles $f, g \in Sym(X)$ are <u>disjoint</u> if

$f(x) \neq x \Rightarrow g(x) = x$ and

$g(x) \neq x \Rightarrow f(x) = x$.

<u>Example</u> $(12), (35) \in S_5$ are disjoint.

$f = (12), g = (123) \in S_3$ are <u>not</u> disjoint

since $f(2) \neq 2$ and $g(2) \neq 2$ either.

<u>Aside</u> $Sym(X)$ acts on X. Suppose $f = (x_1 \ldots x_r) \in Sym(X)$ is a cycle. The subgroup $\langle f \rangle$ acts on X and

$\langle f \rangle \cdot x_1 = \{ x_1, \ldots x_r \}$ since $x_i = f^{i} \cdot x_1$ for $i = 1, \ldots r-1$,

$\langle f \rangle \cdot x = \{ x \}$ if $x \notin \{ x_1, \ldots x_r \}$.

$f \cdot x_r = x_1$ and $f^r = id$.

Theorem (1) Any two disjoint cycles $f, g \in S_n$ commute:
$$fg = gf$$
(2) Any permutation $\sigma \in S_n$ is a product of disjoint cycles; the product is unique up to order.

< proof next time >

It's easy to multiply cycles. For example

$$(12)\circ(235) = \begin{pmatrix} 1 \mapsto 1 \mapsto 2 \\ 2 \mapsto 3 \mapsto 3 \\ 3 \mapsto 5 \mapsto 5 \\ 4 \mapsto 4 \mapsto 4 \\ 5 \mapsto 2 \mapsto 1 \end{pmatrix} = (1235)(4) = (1235)$$

since $(4) = id$

Note $$(235)\circ(12) = \begin{pmatrix} 1 \mapsto 2 \mapsto 3 \\ 3 \mapsto 3 \mapsto 5 \\ 5 \mapsto 5 \mapsto 2 \\ 2 \mapsto 1 \mapsto 1 \\ 4 \mapsto 4 \mapsto 4 \end{pmatrix} = (1352)$$

So $$(12)(235) \neq (235)(12)$$

Definition Let $G \times X \to X$ be an action. The stabilizer of $x \in X$ is
$$G_x \equiv \mathrm{Stab}(x) = \{ g \in G \mid g \cdot x = x \}$$

Ex $GL(3,\mathbb{R})$ acts on $\mathbb{R}^3$. $\mathrm{Stab}(\vec{0}) = GL(3,\mathbb{R})$
$$\mathrm{Stab}\begin{pmatrix}1\\0\\0\end{pmatrix} = \left\{ A \in GL(3,\mathbb{R}) \;\middle|\; A = \left(\begin{array}{c|cc} 1 & * & * \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right), \det B \neq 0 \right\}$$