Last time: Given a subgroup $H$ of a group $G$ the right $H$-coset of $a \in G$ is $Ha := \{ha \mid h \in H\}$

the left $H$-coset of $a$ is $aH := \{ah \mid h \in H\}$.

Right cosets are orbits for an action of $H$ on $G$ and equivalence classes of the relation $a \sim b \Leftrightarrow a = hb$ for some $h \in H \Leftrightarrow ab^{-1} \in H$

Left cosets are orbits for a different action of $H$ on $G$ and are equivalence class of the relation

$$a \sim b \Leftrightarrow a = bh \text{ for some } h \in H \Leftrightarrow b^{-1}a \in H.$$

$H \backslash G$ = the set of all right $H$-cosets

$G/H$ = the set of all left $H$-cosets

We proved: Thm 8.4 A homomorphism $f: G \to H$ induces a well-defined injective map $\bar{f}: K\backslash G \to H$, $\bar{f}(Kg) = f(g)$ where $K := \ker f$.

Hence $\bar{f}: K\backslash G \to \operatorname{im} f$ is a bijection.

[We'll prove soon $K\backslash G$ is naturally a group and $\bar{f}: K\backslash G \to \operatorname{im} f$ is an isomorphism

We also proved:

- If $H$ is a subgroup of $\mathbb{Z}$ then $H = n\mathbb{Z}$ for some $n \geq 0$
- For any group $G$ and any $a \in G$ $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of $G$ which is either isomorphic to $\mathbb{Z}$ (and then the order of $a$ is $\infty$) or it's isomorphic to $\mathbb{Z}_n$ for some $n > 0$. (And then the order of $a$ is $n$).

We need more examples of groups.

The orthogonal group $O(n)$, $n \geq 2$.

$O(n) := \{A \in M_n(\mathbb{R}) \mid A^T A = I\}$

$O(n)$ is a subgroup of $GL(n,\mathbb{R})$:

Check $\forall A \in O(n)$, $1 = \det(I) = \det(A^T A) = \det A^T \det A = (\det A)^2$

$\Rightarrow \det A = \pm 1$ and $A$ is invertible.

$\Rightarrow A^{-1} = I A^{-1} = A^T A A^{-1} = A^T$.

Recall $\emptyset \neq H \leq G$ is a subgroup $\Leftrightarrow \forall a, b \in H$, $ab^{-1} \in H$.

Now for $A, B \in O(n)$

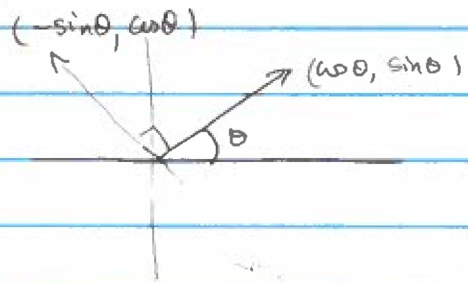
$$(AB^{-1})^T (AB^{-1}) = (AB^T)^T AB^T = (B^T)^T A^T A B^T$$

$$= B I B^T = B B^{-1} = I.$$

$\therefore$ $O(n)$ is a subgroup of $GL(n,\mathbb{R})$, hence a group.

Exercise $SO(n) := \{A \in O(n) \mid \det A = 1\}$ is a subgroup of $O(n)$.

n=2




$$\det\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\Downarrow$$

$$SO(2) = \left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \,\middle|\, \theta \in \mathbb{R}\right\}$$

$\tau := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in O(2) \setminus SO(2)$ since $\det \tau = -1$ (and $\tau^2 = I$)

If $B \in O(2) \setminus SO(2)$, ie $B \in O(2)$ and $\det B = -1$

Then $\det(\tau B) = (-1)(-1) = 1 \Rightarrow \tau B \in SO(2)$

$\Rightarrow B \in \tau\, SO(2)$

$$\boxed{\therefore\ O(2) = SO(2) \cup \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} SO(2).}$$

Note The map $f : SO(2) \to U(1)$, $f\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = e^{i\theta}$

is an isomorphism of groups

Under the identification $\mathbb{R}^2 \longleftrightarrow \mathbb{C}$, $\tau$ corresponds to

$\tau(z) = \bar{z}$, complex conjugation.

**Lemma 9.1** Let $\{H_\alpha\}_{\alpha\in A}$ be a collection of subgroups of a group $G$.
Then $\bigcap_{\alpha\in A} H_\alpha$ is also a subgroup.

**Proof** We need to check that $\bigcap_{\alpha\in A} H_\alpha \neq \emptyset$ and that $a,b\in\bigcap_{\alpha\in A} H_\alpha \Rightarrow ab^{-1}\in\bigcap_{\alpha\in A} H_\alpha$.
Now $e_G \in H_\alpha \ \forall \alpha$.
$\Rightarrow e_G \in \bigcap_{\alpha\in A} H_\alpha$ so $\bigcap H_\alpha \neq \emptyset$.
Also $\forall\, a,b\in \bigcap_{\alpha\in A} H_\alpha$, $a,b\in H_\alpha\ \forall\alpha \Rightarrow ab^{-1}\in H_\alpha\ \forall\alpha \Rightarrow$
$ab^{-1}\in \bigcap_{\alpha\in A} H_\alpha$ $\square$

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**Definition** Let $G$ be a group and $S\subset G$ a set. We define
$$\langle S\rangle = \bigcap_{\substack{H<G\\ S\subseteq H}} H$$
("$H<G$" means "$H$ is a subgroup of $G$").

By 9.1, $\langle S\rangle$ is a subgroup of $G$. We call $\langle S\rangle$
the <u>subgroup of $G$ generated by the set $S$</u>

**Note!** If $K\subseteq G$ is any subgroup with $S\subseteq K$ then $\bigcap_{\substack{H<G\\ S\subseteq H}} H \subseteq K$

So

$\langle S\rangle$ is the smallest subgroup of $G$ that contains $S$.

**Ex** $G$ any group, $g\in G$ and $S=\{g\}$.
Then $\langle\{g\}\rangle = \{g^n \mid n\in\mathbb{Z}\} = \langle g\rangle$.

**Ex** Let $G=O(2) = SO(2)\cup\tau SO(2) = U(1)\cup \tau U(1)$
$\tau$ ↑ complex conj.
Fix $n>1$ and consider
$$\rho = e^{2\pi i/n}\in U(1)$$

**Claim** $D_n := \langle\{\rho,\tau\}\rangle = \{1,\rho,\rho^2,\dots,\rho^{n-1},\tau,\tau\rho,\dots,\tau\rho^{n-1}\}$.
$D_n$ is called the dihedral group of order $n$.
Since $\rho^n = (e^{2\pi i/n})^n = e^{2\pi i} = 1$, $\langle\rho\rangle = \{1,\rho,\dots,\rho^{n-1}\}\subseteq D_n$.
And then $\tau\rho^k\in D_n$ for $k=0,1,\dots,n-1$ since $\tau,\rho^k\in D_n$

Note also that $\tau(\tau(z)) = \overline{\overline{z}} = z \Rightarrow \tau^2 = id. \Rightarrow \boxed{\tau = \tau^{-1}}$

And

$$\tau(\rho(z)) = \tau(e^{2\pi i/n} z) = e^{-2\pi i/n}\,\overline{z} = \rho^{-1}\tau(z)$$

$$\therefore \boxed{\tau\rho = \rho^{-1}\tau} \quad (= \rho^{n-1}\tau)$$

(or, equivalently $\tau\rho\tau^{-1} = \rho^{-1}$)

Induction on $k$ $\Rightarrow$

$$\tau\rho^k\tau^{-1} = \underbrace{\tau\rho\tau^{-1}\tau\rho\tau^{-1}\cdots\tau\rho\tau^{-1}}_{k} = (\rho^{-1})^k = \rho^{-k}.$$

Hence 1) $\tau\rho^k = \rho^{-k}\tau \quad \forall k$

2) $(\tau\rho^k)^{-1} = (\rho^k)^{-1}\tau^{-1} = \rho^{-k}\tau = \tau\rho^k.$

3) $(\tau\rho^k)(\tau\rho^j) = (\tau\rho^k\tau^{-1})\rho^j = \rho^{-k}\rho^j = \rho^{j-k}$

$\therefore \{1, \rho, \ldots \rho^{n-1}, \tau\rho, \ldots \tau\rho^{n-1}\}$ is a subgroup of $O(2)$ that contains $\{\tau, \rho\}$.

$$\Rightarrow \langle\{\tau,\rho\}\rangle = \{1, \rho, \ldots \rho^{n-1}, \tau\rho, \ldots \tau\rho^{n-1}\}$$

Note that multiplication of elements of $D_n$ follows 3 rules:

$$\rho^n = e, \quad \tau^2 = e, \quad \tau\rho = \rho^{-1}\tau$$

One writes $D_n = \langle \tau, \rho \mid \rho^n = \tau^2 = e,\ \tau\rho = \rho^{-1}\tau \rangle$

This is a presentation of $D_n$ by <u>generators</u> and <u>relations</u>