Last time:
- isomorphisms of groups; subgroups
- criterion for $\emptyset \neq H \subset G$ to be a subgroup
- $\ker(f : G \to H)$, $\operatorname{im}(f : G \to H)$ are subgroups.
- $f : G \to H$ is 1-1 $\Leftrightarrow$ $\ker f = \{e_G\}$.
- $\forall a \in G$ $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G.

$\langle a \rangle \simeq \mathbb{Z}$ ("$\simeq$" means "isomorphic") $\Leftrightarrow$ $(a^m = e_G \Rightarrow m = 0)$

We'll see later: if $\exists\, n \neq 0$ s.t. $a^n = e$ then $\langle a \rangle \simeq \mathbb{Z}_m$ for some $m > 1$, $m \leq |n|$.

Today Group actions (Nicholson, § 8.3)

Def 7.1 An *action* of a group G on a set X is a map (a function)

$G \times X \to X$, $(g, x) \mapsto g \cdot x$ so that

i) $e_G \cdot x = x$ $\forall x \in X$

ii) $\underbrace{a \cdot (b \cdot x)}_{\text{action}} = \underbrace{(ab)}_{\text{mult in } G} \cdot_{\text{action}}\, x$

Examples I. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$: $\forall A \in GL(n, \mathbb{R})$ $\forall v \in \mathbb{R}^n$

$$A \cdot v := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$ a matrix acting on a vector.

It's an action since $\forall v \in \mathbb{R}^n$, 1) $I \cdot v = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v$

2) $\forall A, B \in GL(n, \mathbb{R})$

$$A \cdot (B \cdot v) = A(Bv) = \underbrace{(AB)}_{\text{matrix mult.}} v = (AB) \cdot v$$

II. $G = U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ acts on $\mathbb{C}$ by

$$\lambda \cdot z := \lambda z \quad \text{mult of complex numbers.}$$

Then i) $1 \cdot z = 1z = z$

ii) $\lambda \cdot (\mu \cdot z) = \lambda(\mu z) = (\lambda\mu) z = (\lambda\mu) \cdot z$

III Def. X be a set. $G = Sym(X) = \{f : X \to X \mid f \text{ is a bijection}\}$

Recall: $Sym(X)$ is a group w. group operation composition and

$$e_{Sym(X)} = id_X : X \to X, \quad id_X(x) = x$$

$Sym(X)$ acts on X by

$$f \cdot x := f(x)$$

It's an action since $id_X \cdot x = id_X(x) = x$ and $\forall f, g \in Sym(X)$

$$f \cdot (g \cdot x) = f(g(x)) = (f \circ g)(x) = (f \circ g) \cdot x.$$

IV A group G acts on itself by multiplication on the left:

$$G \times G \to G, \quad g \cdot x := gx$$

Check $e_G \cdot x = e_G x = x \quad \forall x \in G$

$$a \cdot (b \cdot x) = a(bx) = (ab)x = (ab) \cdot x \quad \forall a, b \in G, x \in G.$$

Non-example Suppose G is a non-abelian group

ie. $\exists a, b \in G$ s.t. $ab \neq ba$

Then the multiplication on the right

$$G \times G \to G, \ (g, x) \mapsto xg \quad (\text{ie } g \cdot x = xg)$$

is not an action.

Reason if $ab \neq ba$ $\quad (ab) \cdot x = x(ab)$

while $a \cdot (b \cdot x) = (xb)a$

and $xab \neq xba$ if $ab \neq ba$.

This problem can be fixed:

$$G \times G \to G, \quad g \cdot x := xg^{-1} \text{ is an action}$$

This is because i) $e^{-1} = e$ so $xe^{-1} = xe = x$

ii) $\forall a, b \in G \quad (ab)^{-1} = b^{-1}a^{-1}$ since

$$(b^{-1}a^{-1})ab = b^{-1}(a^{-1}a)b = b^{-1}eb = e$$

and then $\quad ab \cdot x = x(ab)^{-1} = (xb^{-1})a^{-1} = a \cdot (b \cdot x)$

Ex Any group G acts on itself by conjugation:

$$g \cdot x := g x g^{-1} \quad \text{is an action.}$$

Check: i) $e \cdot x = e x e^{-1} = x$

ii) $ab \cdot x = ab\, x\, (ab)^{-1} = a b x b^{-1} a^{-1} = a \cdot (b \cdot x)$

Definition A action of a group G on a set X is trivial if

$$g \cdot x = x \quad \forall g \in G \quad \forall x \in X.$$

(It's ~~is~~ an action. Check it.)

Remark if a group G acts on a set X then a subgroup H of G also acts on X: $\forall h \in H$ $h \cdot x$ makes sense since $h \in G$.
and the multiplication on H is inherited from the one on G, so

$$h_1 \cdot (h_2 \cdot x) = (h_1 h_2) \cdot x \quad \forall h_1, h_2 \in H.$$

Definition (orbits) Let $G \times X \to G$ be an action of a group G on a set X, $x_0 \in X$ a point.

The orbit of x_0 is the set

$$G \cdot x_0 := \{ g \cdot x_0 \mid g \in G \}$$

Ex $U(1)$ acts on $\mathbb{C}$ by $\lambda \cdot z = \lambda z$

Claim 1 $\quad U(1) \cdot 0 = \{0\}$

Claim 2 $\quad U(1) \cdot 2 = \{ z \in \mathbb{C} \mid |z| = 2 \}$

Reason $\forall \lambda \in U(1) \quad \lambda \cdot 0 = \lambda 0 = 0 \Rightarrow U(1) \cdot 0 = \{0\}$.

$\forall \lambda \in U(1) \quad |\lambda \cdot 2| = |\lambda 2| = |\lambda| |2| = 1 \cdot 2 = 2$

$\Rightarrow \lambda \cdot 2 \in \{ z \in \mathbb{C} \mid |z| = 2 \}$

Conversely $\forall w \in \{ z \in \mathbb{C} \mid |z| = 2 \}$

$|\frac{1}{2} w| = \frac{1}{2} |w| = \frac{1}{2} \cdot 2 = 1$ and $(\frac{1}{2} w) \cdot 2 = \frac{1}{2} w \cdot 2 = w$.

$\Rightarrow \{ z \in \mathbb{C} \mid |z| = 2 \} \subseteq U(1) \cdot 2$ as well

In general $U(1) \cdot z = \{ w \in \mathbb{C} \mid |w| = |z| \}$ check it.

Theorem 7.1 Let $G \times X \to X$ be an action.
Then the set of all G-orbits $\{G\cdot x\}_{x\in X}$ is a partition X.

(The corresponding (equivalence) relation is: $x \sim x' \iff \exists g \in G$ s.t. $g\cdot x = x'$.)

Proof 1) $\forall x \in X$, $e\cdot x = x$. $\Rightarrow x \in G\cdot x$

$\Rightarrow X \subseteq \bigcup_{x\in X} G\cdot x \; (\subseteq X)$. $\Rightarrow X = \bigcup_{x\in X} G\cdot x$

2) Suppose $G\cdot x \cap G\cdot y \neq \emptyset$. Then $\exists z \in X$ s.t. $z \in G\cdot x \cap G\cdot y$

$\Rightarrow \exists a, b \in G$ s.t. $a\cdot x = z = b\cdot y$

$\Rightarrow x = e\cdot x = (a^{-1}a)\cdot x = a^{-1}\cdot(a\cdot x) = a^{-1}\cdot(b\cdot y)$

$= (a^{-1}b)\cdot y$

$\Rightarrow \forall g \in G$. $g\cdot x = g\cdot\big((a^{-1}b)\cdot y\big) = (g a^{-1} b)\cdot y \in G\cdot y$

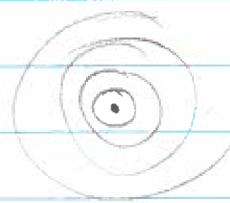
$\Rightarrow G\cdot x \subseteq G\cdot y$.

Similarly $G\cdot y \subseteq G\cdot x$.

$\therefore G\cdot x \cap G\cdot y \neq \emptyset \Rightarrow G\cdot x = G\cdot y$

ie $\{G\cdot x\}_{x\in X}$ is a partition.

Picture for the action of $U(1)$ on $\mathbb{C}$:



Ex $GL(2,\mathbb{R})$ acts on $\mathbb{R}^2$.

$\forall v \in \mathbb{R}^2$, $v \neq \vec{0}$ $\exists w \in \mathbb{R}^2$ s.t. $\{v, w\}$ is a basis.

But then $A = (v\,|\,w) \in GL(2,\mathbb{R})$ and $A\begin{pmatrix}1\\0\end{pmatrix} = v$.

$\Rightarrow$ There are only two orbits: $\{\vec{0}\}$ and $GL(2,\mathbb{R})\cdot\begin{pmatrix}1\\0\end{pmatrix}$.