"Review" of vector spaces

Def (Goodman, 3.3.1) A vector space V over a field F is an abelian group $(V, +, 0)$ together with a map

$$F \times V \longrightarrow V, \quad (\lambda, v) \longmapsto \lambda v \qquad \text{"multiplication by scalars"}$$

so that

(a) $1v = v \quad \forall v \in V$

(b) $\underset{\text{mult. in } K}{(\alpha\beta)} v = \alpha \underbrace{(\beta v)}_{\text{scalar mult.}} \quad \forall \alpha, \beta \in F \quad \forall v \in V$

(c) $\alpha(v + w) = (\alpha v) + (\alpha w) \quad \forall \alpha \in F, \ v, w \in V$

(d) $(\alpha + \beta) v = (\alpha v) + (\beta v) \quad \forall \alpha, \beta \in F \quad v \in V.$

Examples (1) For any field F any $n \geq 0$

$$F^n = \{(x_1, \ldots, x_n) \mid x_i \in F\} \text{ is a vector space over } F$$

The scalar multiplication is

$$\lambda(x_1 \ldots x_n) := (\lambda x_1, \ldots \lambda x_n)$$

(2) For any field F, $F[x]$ is a vector space over F:

$$\lambda \cdot \left(\sum_{i=0}^{n} a_i x^i \right) := \sum (\lambda a_i) x^i \qquad \forall \lambda \in F, \ \forall \sum a_i x^i \in F[x]$$

(3) For any $n \geq 1$

$$F[x]_{\leq n} := \{ p \in F[x] \mid \deg p \leq n \}$$

is a vector space over F.

Vector spaces have bases and dimensions.

Recall

Def Let V be a vector space over a field F.

A subset $\{v_1, \dots v_n\} \subseteq V$ spans V if $\forall w \in V$
$\exists \lambda_1, \dots \lambda_n \in F$ s.t. $w = \sum \lambda_i v_i$
($\lambda_1, \dots \lambda_n$ depend on w)

Ex $\{1, x, \dots x^n\}$ spans $F[x]_{\le n}$:
if $p(x) \in F[x]_{\le n}$ $\exists a_0, \dots a_n \in F$ s.t. $p(x) = \sum_{i=0}^{n} a_i x^i$.

Ex $\{(1,0), (0,1), (1,1)\}$ spans F^2 (for any field F)

$$(x_1, x_2) = x_1 (1,0) + x_2 (0,1) + 0 \cdot (1,1)$$
$$= 0 \cdot (1,0) + x_1 (1,1) + (x_2 - x_1) \cdot (0,1).$$

Definition A subset $\{v_1, \dots v_k\}$ of a vector space V is linearly independent if for all $c_1, \dots, c_k \in V$

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = \vec{0} \Rightarrow c_1 = c_2 = \dots = c_k = 0$$

Ex $\{1, x, \dots x^n\} \subseteq F[x]_{\le n}$ is linearly independent:

$$c_0 \cdot 1 + c_1 x^1 + c_2 x^2 + \dots + c_k x^k = 0$$
$$\Leftrightarrow c_0 = c_1 = \dots = c_k = 0.$$

Definition A subset $\{v_1, \dots v_k\}$ of a vector space V over a field F is a basis iff it's linearly independent and spans V.

Ex $\{(1,0), (0,1)\}$ is a basis of F^2
$\{(1,0), (1,1), (0,1)\}$ is not a basis of F^2
$\{1, x, \dots x^n\}$ is a basis of $F[x]_{\le n}$.

Fact $\{v_1, \dots v_k\}$ is a basis of $V \Leftrightarrow \forall w \in V$

there are unique $c_1, \ldots, c_n \in F$ s.t. $w = \sum c_i v_i$.

Definition A vector space V is finite-dimensional $\Leftrightarrow$ V is spanned by a finite set.

Ex $F[x]_{\leq n}$ is finite dimensional v.space $/F$
$F[x]$ is not a finite dim. vector space over F.

Theorem (i) Any finite dimensional vector space has a basis
(ii) Any two different bases of a finite dimensional vector space have the same number of elements.
This number is called the dimension of the vector space.

Ex $\dim(F[x]_{\leq n}) = n+1$.

Proposition 34.1 Let F be a field, $f(x) \in F[x]$ a polynomial of degree $n>0$. Then the ring $F[x]/\langle f\rangle$ is a vector space over F of dimension n.

Proof (1) Note first that since $F[x]/\langle f\rangle$ is a ring, it is, in particular an abelian group.

(2) Next note that the ring homomorphism $\varphi: F \to F[x]/\langle f\rangle$, $\varphi(\lambda) = \lambda + \langle f\rangle$, is injective. This is because $a \in \ker\varphi \Leftrightarrow a + \langle f\rangle = 0 + \langle f\rangle$
$\Leftrightarrow a \in \langle f\rangle \Leftrightarrow a = fh$ for some $h \in F[x]$.

Now $0 \geq \deg a = \deg(fh) = \deg f + \deg h = n + \deg h$.
Since $n>0$, this is only possible if $\deg a = \deg h = -\infty$, ie
$a = h = 0$. $\Rightarrow \ker\varphi = \{0\}$, $\Rightarrow$ φ is injective

We define scalar multiplication $F \times F[x]/\langle f\rangle \to F[x]/\langle f\rangle$ by
$\lambda \cdot (p + \langle f\rangle) := \varphi(\lambda)\,(p + \langle f\rangle)$.

Note that $(\lambda + \langle f \rangle)(p + \langle f \rangle) = \lambda p + \langle f \rangle$.

It's not hard to check that this scalar multiplication makes $F[x]/\langle f \rangle$ into a vector space over F.

Claim The set $\{1 + \langle f \rangle, x + \langle f \rangle, \ldots, x^{n-1} + \langle f \rangle\}$ spans $F[x]/\langle f \rangle$.

Proof Given $p + \langle f \rangle \in F[x]$ $\exists\, q, r \in F[x]$ s.t.

(i) $p = qf + r$ and (ii) $\deg r < \deg f = n$.

(i) $\Rightarrow$ $p - r \in \langle f \rangle \Rightarrow p + \langle f \rangle = r + \langle f \rangle$

(ii) $\Rightarrow$ $r = c_0 x^0 + c_1 x^1 + \ldots + c_{n-1} x^{n-1}$ for some $c_0, \ldots, c_{n-1} \in F$

$\Rightarrow$ $p + \langle f \rangle = \left(\sum_{i=0}^{n-1} c_i x^i\right) + \langle f \rangle = \sum_{i=0}^{n-1} c_i \left(x^i + \langle f \rangle\right)$

Claim $\{x^0 + \langle f \rangle, \ldots, x^{n-1} + \langle f \rangle\}$ is linearly independent

Proof Suppose $\exists\, c_0, \ldots, c_{n-1} \in F$ s.t.

$$\sum_{i=0}^{n-1} c_i (x^i + \langle f \rangle) = 0 + \langle f \rangle.$$

Then $\langle f \rangle = (\sum c_i x^i) + \langle f \rangle. \Rightarrow \sum_{i=0}^{n-1} c_i x^i \in \langle f \rangle$.

$\Rightarrow$ $\sum_{i=0}^{n-1} c_i x^i = f k$ for some $k(x) \in F[x]$

Degree argument again $\Rightarrow$ $k = 0$ and $\sum c_i x^i = 0$ in $F[x]$.

$\Rightarrow$ $c_0 = c_1 = \ldots = c_{n-1} = 0$ $\square$

We conclude that $\{x^i + \langle f \rangle\}_{i=0}^{n-1}$ is a basis of $F[x]/\langle f \rangle$. $\square$