Last time Constructed a homomorphism $\rho : S_n \to GL(n, \mathbb{R})$

$$\rho(\sigma) = (e_{\sigma(1)} | \dots | e_{\sigma(n)})$$

(Hence an action of S_n on $\mathbb{R}^n$ given by

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(n)})\ )$$

Defined

$$\text{sign}(\sigma) = \frac{\Delta(\rho(\sigma)\vec{x})}{\Delta(\vec{x})} \quad \text{where } \Delta(\vec{x}) = \prod_{i<j} (x_i - x_j)$$

Proved

$\text{sign}: S_n \to \{\pm 1\}$ is onto and, in particular,

$\text{sign}((ij)) = -1.$

Argued that $\text{sign}(\tau) = \det(\rho(\tau))$

where $\det((a_{ij})) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$.

Definition Suppose a group G acts on a set X, $x \in X$.
The stabilizer of x is the set $G_x = \{ g \in G \mid g \cdot x = x \}$.

Ex $G = SO(3) = 3\times 3$ matrices s.t. $A^T A = I$ & $\det(A) = +1$.
$SO(3)$ acts on $\mathbb{R}^3$: $A \cdot \vec{v} = A\vec{v}$.
matrix acting on a vector.

$\text{Stab}(\vec{0}) = SO(3)$ since $A\vec{0} = \vec{0}$ $\forall A \in SO(3)$
If $\vec{v} \neq 0$, $\text{Stab}(\vec{v})$ = rotations about the line through $\vec{v}$.

Lemma 5.1.12 (in Goodman) The stabilizer G_x is a subgroup of G.

Proof (i) $\forall a, b \in G_x$

$$\begin{aligned}(ab)\cdot x = a \cdot (b \cdot x) &= a \cdot x \quad (\text{since } b \in G_x)\\ &= x \quad (\text{since } a \in G_x)\end{aligned}$$

$\Rightarrow ab \in G_x$. $\Rightarrow G_x$ is closed under multiplication.

(ii) $e \cdot x = x \Rightarrow e \in G_x$

(iii) $\forall a \in G_x$, $x = e \cdot x = (a^{-1}a)\cdot x = a^{-1}\cdot(a \cdot x) = a^{-1} \cdot x$

$\therefore \forall a \in G_x$, $a^{-1} \in G_x$.

$\Rightarrow$ G_x is a subgroup. $\square$

<u>Goodman, Proposition 5.1.13</u> Suppose a group G acts on a set X, $x \in X$.
The map $\psi : G/G_x \to G\cdot x = \{g\cdot x \mid g \in G\}$

$$\psi(aG_x) = a\cdot x$$

is a well-defined bijection.

<u>Proof</u> (1) Suppose $aG_x = bG_x$. We need to show: $a\cdot x = b\cdot x$.

$aG_x = bG_x \Rightarrow a = bh$ for some $h \in G_x$.

$\Rightarrow a\cdot x = (bh)\cdot x = b\cdot(h\cdot x) = b\cdot x$ (since $h \in G_x$!)

$\Rightarrow$ ψ is well-defined

(2) Suppose $y \in G\cdot x$. Then $y = a\cdot x$ for some $a \in G$.

$\Rightarrow \psi(aG_x) = a\cdot x = y$. $\Rightarrow$ ψ is onto.

(3) Suppose $\psi(aG_x) = \psi(bG_x)$. Then $a\cdot x = b\cdot x$

$\Rightarrow b^{-1}\cdot(a\cdot x) = b^{-1}\cdot(b\cdot x) = (b^{-1}b)\cdot x = e\cdot x = x$.

$\Rightarrow (b^{-1}a)\cdot x = x$. $\Rightarrow$ $b^{-1}a = h$ for some $h \in G_x$.

$\Rightarrow a = bh$ for some $h \in G_x$

$\Rightarrow aG_x = bG_x$.

$\therefore$ ψ is 1-1.

<u>Corollary 5.1.14</u> in Goodman. Suppose G is a finite group.
Then $|G\cdot x|\cdot|G_x| = |G|$.
In particular $|G_x| \mid |G|$ and $|G\cdot x| \mid |G|$.

<u>Proof</u> Recall Lagrange's theorem: for any subgroup H of G

$$|G| = |G/H|\,|H|$$

$\Rightarrow |G| = |G/G_x|\,|G_x|$.

By 5.1.13 $\exists$ a bijection $G\cdot x \leftrightarrow G/G_x \Rightarrow |G\cdot x| = |G/G_x|$

$\Rightarrow |G| = |G\cdot x|\,|G_x|$. $\square$

Conjugation and conjugacy classes.

Let G be a group. We have a map

$$G \times G \to G, \quad (g,x) \mapsto g \cdot x := g x g^{-1}$$

It's an action: (i) $e \cdot x = e x e^{-1} = x$

(ii) $\forall\, g_1, g_2 \in G$ $\quad g_1 \cdot (g_2 \cdot x) = g_1 (g_2 x g_2^{-1}) g_1^{-1} =$

$$= g_1 g_2\, x\, g_2^{-1} g_1^{-1} = (g_1 g_2) x (g_1 g_2^{-1})$$
$$= (g_1 g_2) \cdot x.$$

This action is called <u>conjugation</u>

Corresponding to this action we have a homomorphism

$$c : G \to \mathrm{Sym}(G), \quad c_g(x) = g x g^{-1}.$$
$$g \mapsto c_g$$

<u>Definition</u> The orbits of conjugation are called <u>conjugacy classes.</u>

Thus, the conjugacy class of $x \in G$ is the set $\mathrm{Conj}(x) := \{ g x g^{-1} \mid g \in G \}$.

<u>Ex</u> Suppose G is <u>abelian</u>: $ab = ba$ $\forall\, a, b \in G$.

Then the conjugacy class $\mathrm{Conj}(x)$ of any $x \in G$ is

$$\mathrm{Conj}(x) = \{ g x g^{-1} \mid g \in G \} = \{ x g g^{-1} \mid g \in G \} = \{x\}$$

<u>Ex</u> $G = S_n$ $\quad \sigma = (a_1 \ldots a_r)$, an r-cycle.

<u>Claim</u> The conjugacy class of an r-cycle σ is the set of <u>all</u> r-cycles.

<u>Proof</u> $\forall\, \tau \in S_n$

$$\tau \sigma \tau^{-1} = \tau (a_1 \ldots a_n) \tau^{-1} \overset{\text{homework!}}{=} (\tau(a_1) \ldots \tau(a_n))$$

If $(b_1 \ldots b_r)$ is any r-cycle, consider $\tau \in S_n$ defined by

$$\tau(a_i) = b_i \qquad 1 \le i \le r$$
$$\tau(j) = j \qquad j \ne a_i \text{ for any } i$$

Then $\tau(a_1 \dots a_r)\tau^{-1} = (\tau(a_1) \dots \tau(a_r)) = (b_1 \dots b_r)$.

Hence $\quad Conj\,((a_1 \dots a_r)) = \{ (b_1 \dots b_r) \mid 1 \le b_i \le n,\ b_i \ne b_j \}$

$= $ the set of all r-cycles.

<u>Question</u> What is the conjugacy class of an arbitrary permutation $\mu \in S_n$?

<u>Remark</u> For any group G, $c_g : G \to G$, $c_g(x) = gxg^{-1}$ is a homomorphism:

$$c_g(x_1 x_2) = g x_1 x_2 g^{-1} = g x g^{-1} g x g^{-1} = c_g(x_1)\, c_g(x_2).$$

Now any $\mu \in S_n$ is a product of disjoint cycles:

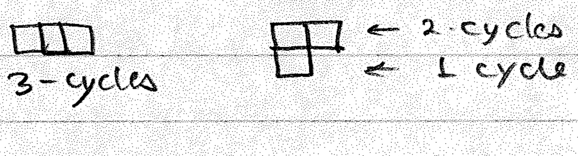
$$\mu = x_1 x_2 \dots x_k \qquad x_i = r_i\text{-cycle for some } r_i$$

with $r_1 + r_2 \dots + r_k = n$

(to have $r_1 + \dots + r_k = n$ we need to include "1-cycles")

The conjugacy class of μ = the set of all possible partitions of $\{1, \dots n\}$ into sets of sizes $r_1, \dots r_k$.

<u>Ex</u> <u>$n=3$</u> The possible partitions are



3 1-cycles, ie id.

<u>Ex</u> $n=4$