10/15/201
Mo

Recall An element $\tau \in S_n$ is a transposition if τ is a 2-cycle: $\tau = (ij)$ for some $i,j \in \{1, \dots, n\}$.

Goal Construct a homomorphism $\text{sign}: S_n \to \{\pm 1\}$.
so that $\text{sign}(ij) = -1 \quad \forall (i,j) \in S_n$.

Aside (i) $A_n = \ker(\text{sign})$ is a normal subgroup of S_n called the alternating group.
(ii) Since sign is onto, $S_n / A_n \cong \{\pm 1\}$.

Key idea: S_n acts on the standard basis $\{e_1, \dots, e_n\}$ by permuting the vectors: for $\sigma \in S_n$, $\sigma \cdot e_i = e_{\sigma(i)}$.
This action gives rise to a collection of linear maps

$$\rho(\sigma): \mathbb{R}^n \to \mathbb{R}^n, \quad \rho(\sigma)\left(\sum x_i e_i\right) := \sum x_i e_{\sigma(i)}$$

Each $\rho(\sigma)$ is a matrix with the columns $e_{\sigma(i)}$'s:

$$\rho(\sigma) = \left(e_{\sigma(1)} \middle| e_{\sigma(2)} \middle| \dots \middle| e_{\sigma(n)} \right)$$

Ex $n = 3$, $\sigma = (1\,2\,3)$

$$\rho(\sigma) = \left(e_{\sigma(1)} \middle| e_{\sigma(2)} \middle| e_{\sigma(3)} \right) = \left(e_2 \middle| e_3 \middle| e_1 \right) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Since the columns of $\rho(\sigma)$ are basis vectors, each $\rho(\sigma)$ is an invertible matrix, that is, an element of
$GL(n, \mathbb{R})$ = the group of $n \times n$ invertible matrices

The map $\rho: S_n \to GL(n, \mathbb{R})$ is a homomorphism:
$\forall \sigma, \mu \in S_n \quad \forall \vec{x} = \sum x_i e_i \in \mathbb{R}^n$

$$\rho(\sigma)(\rho(\mu)\vec{x}) = \rho(\sigma)\left(\sum_i x_i e_{\mu(i)}\right) = \sum_i x_i e_{\sigma(\mu(i))} =$$

$= \sum_i x_i e_{(\sigma\mu)(i)} = \rho(\sigma\mu)(\sum x_i e_i).$

<u>Note</u> Since $\sum_{i=1}^n x_i e_{\sigma(i)} = \begin{pmatrix} j=\sigma(i) \\ i=\sigma^{-1}(j) \end{pmatrix} = \sum_{j=1}^n x_{\sigma^{-1}(j)} e_j$

$$\rho(\sigma)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}$$

For $\sigma \in S_n$, $\rho(\sigma)$ is called the corresponding <u>permutation matrix</u>

The homomorphism $\rho: S_n \to GL(n, \mathbb{R})$ is called the permutation representation: we "represent" permutations by matrices.

<u>Note</u> ρ is injective since $\sigma \in \ker \rho \Leftrightarrow$

$$\rho(\sigma) = I$$

$$\Leftrightarrow \left(e_{\sigma(1)} | \dots | e_{\sigma(n)}\right) = \left(e_1 | e_2 | \dots | e_n\right)$$

$$\Leftrightarrow \sigma = id.$$

Homomorphism theorem $\Rightarrow \rho(S_n) \subseteq GL(n, \mathbb{R})$ is a subgroup isomorphic to S_n, ie each $\sigma \in S_n$ is represented by a <u>unique</u> matrix $\rho(\sigma)$.

Back to sign: $S_n \to GL(n, \mathbb{R})$

Consider a function $\Delta: \mathbb{R}^n \to \mathbb{R}$,

$$\Delta(x_1 \dots x_n) = \prod_{i<j}(x_i - x_j)$$

<u>Ex</u> $n=3$ $\quad \Delta(x_1, x_2, x_3) = (x_1-x_2)(x_1-x_3)(x_2-x_3)$

<u>$n=4$</u> $\quad \Delta(x_1, x_2, x_3, x_4) = (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)$

etc

[Exercise 19.0 $\Delta\big(\rho(i\ i{+}1)(x_1 - x_n)\big) = -\Delta(x_1, \ldots, x_n)$

Eg $\Delta\big(\rho(34)(x_1 - x_4)\big) = (x_1 - x_2)(x_1 - x_4)(x_1 - x_3)(x_2 - x_4)(x_2 - x_3)(x_4 - x_3)$
$= -\Delta(x_1, - x_4)$.

In general, $\forall \sigma \in S_n$ $\boxed{\Delta\big(\rho(\sigma)(x_1 \ldots x_n)\big) = \pm\Delta(x_1, \ldots x_n)}$

Definition We define $\text{sign}: S_n \to \pm 1$ by

$$\text{sign}(\sigma) = \frac{\Delta\big(\rho(\sigma)(x_1 \ldots x_n)\big)}{\Delta(x_1, \ldots x_n)}$$

Proposition 19.1 For any $\sigma, \mu \in S_n$

$$\text{sign}(\sigma\mu) = \text{sign}(\sigma)\,\text{sign}(\mu),$$

i.e. $\text{sign}: S_n \to \{\pm 1\}$ is a homomorphism.

Proof $$\text{sign}(\sigma\mu) = \frac{\Delta(\rho(\sigma\mu)\vec{x})}{\Delta(\vec{x})} = \frac{\Delta(\rho(\sigma)\rho(\mu)\vec{x})}{\Delta(\rho(\mu)\vec{x})} \cdot \frac{\Delta(\rho(\mu)\vec{x})}{\Delta(\vec{x})}$$

$$= \frac{\Delta(\rho(\sigma)\vec{y})}{\Delta(\vec{y})} \cdot \frac{\Delta(\rho(\mu)\vec{x})}{\Delta(\vec{x})} = \text{sign}(\sigma)\cdot\text{sign}(\rho)$$ $\square$

Note exercise 19.0 says: $\text{sign}(i\ i{+}1) = -1$.)

Lemma 19.2 Any transposition $(i j)$ is a product of an odd number of transpositions of the form $(k\ k{+}1)$. Hence

$$\text{sign}(ij) = (-1)^{\text{odd power}} = -1.$$

Proof Say $i < j$ $\qquad - - \sqcup\, i \underbrace{\sqcup \sqcup \ldots \sqcup}_{j-i-1 \text{ slots}} j \sqcup \ldots$

To switch i and j we first move i past j in

$i \leftrightarrow i{+}1 \leftrightarrow i{+}2 \ldots \leftrightarrow j$ $\quad$ ($j-i$ switches)

Then move j back to where i was

$i{-}1 \quad i{+}1 \ldots \leftrightarrow j{-}1 \leftrightarrow j\ i$ $\quad$ ($j-i-1$ switches)

$\Rightarrow$ total # of switches is $2(j-i)+1$. $\square$

<u>Aside</u> 1) For an $n\times n$ matrix $(a_{ij})_{1\le i,j\le n}$

(*) $\det(a_{ij}) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$.

2) $\det(\rho(\sigma)) = \det(e_{\sigma(1)} | \cdots | e_{\sigma(n)}) = \mathrm{sign}(\sigma)\cdot 1\cdot 1 \cdots \cdot 1$
$= \mathrm{sign}(\sigma)$.

If $\det : GL(n,\mathbb{R}) \to \mathbb{R}$ is constructed in a way that <u>does not</u> use (*) above, we could use it to construct sign:

$$\mathrm{sign} = \det \circ \rho$$

where $\rho : S_n \to GL(n,\mathbb{R})$ is the permutation representation.

<u>More definitions</u>

Def (5.1.11 of Goodman) Suppose a group G acts on a set X. The <u>stabilizer</u> of $x\in X$ is the set

$$G_x = \mathrm{Stab}(x) = \{ g\in G \mid g\cdot x = x\}.$$

<u>Ex</u> $SO(3)$ = 3×3 orthogonal matrices w. $\det = 1$
(the group of <u>proper</u> rotations)
$SO(3)$ acts on $\mathbb{R}^3$: $\quad A\cdot\vec{v} := A\vec{v} \quad \forall A\in SO(3)\ \forall \vec{v}\in\mathbb{R}^3$
$\mathrm{Stab}(\vec{0}) = SO(3)$ since $A\vec{0} = \vec{0} \quad \forall A$.
If $\vec{v}\neq 0$, $\mathrm{Stab}(\vec{v})$ = rotations about the axis $\mathbb{R}\vec{v}$.

<u>Lemma 5.1.12</u> The stabilizer G_x of x is a subgroup of G.

<u>Proof</u> (1) $\forall a,b\in G_x \quad (ab)\cdot x = a\cdot(b\cdot x) = a\cdot x = x \Rightarrow ab\in G_x$.
(2) $e\cdot x = x \Rightarrow e\in G_x$
(3) $\forall a\in G_x$, $\quad x = e\cdot x = (a^{-1}a)\cdot x = (a^{-1})\cdot(a\cdot x) = a^{-1}\cdot x$
$\Rightarrow a^{-1}\in G_x$
$\therefore$ G_x is a subgroup of G.