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Fr
15.1

Review of complex numbers.

Informally, a complex number z is an expression $a+i\cdot b$, where $a, b \in \mathbb{R}$ and $i^2 = i \cdot i = -1$.

We assume: i commutes with real numbers : $i \cdot b = b \cdot i \ \forall b \in \mathbb{R}$

We add and multiply complex numbers as follows:

1) $(a+ib)+(c+id) = (a+c) + i(b+d) \qquad \forall a,b,c,d \in \mathbb{R}$

2) $(a+ib)\cdot(c+id) = ac + aid + ibc + ibid =$
$$= (ac + i^2 bd) + i(ad+bc)$$
$$= (ac - bd) + i(ad+bc)$$

We now turn these informal ideas into a definition:

Def The ring $\mathbb{C}$ of complex numbers is $\mathbb{R}^2$ (ordered pairs of real numbers) with addition $+$ defined coordinate-wise
$$(a,b) + (c,d) = (a+c, b+d)$$
The multiplication $\cdot$ is defined by
$$(a,b)\cdot(c,d) = (ac-bd, ad+bc)$$
One checks that $+$ and $\cdot$ are associative and that $\cdot$ distributes over $+$.

Since $(0,0)+(a,b) = (a,b)$, $0_{\mathbb{C}} = (0,0)$

Since $(1,0)\cdot(a,b) = (1\cdot a - 0\cdot b, 1\cdot b + 0\cdot a) = (a,b)$

$1_{\mathbb{C}} = (1,0)$.

Also, $(a,b) + (-a,-b) = (0,0)$

So $-(a,b) = (-a,-b)$

$+$ has additive inverses.

Note $(0,1)\cdot(0,1) = (0 - 1\cdot 1, 0\cdot 1 + 1\cdot 0) = (-1,0) = -1_{\mathbb{C}}$.

So we set $i = (0,1)$

We identify $\mathbb{R}$ with $\{(a,b)\in\mathbb{C} \mid b=0\}$.

Then $(a,b) = (a,0) + (0,b) = (a,0)\cdot 1_{\mathbb{C}} + (b,0)\cdot(0,1)$

$= (a,0)\cdot 1_{\mathbb{C}} + i\cdot(b,0)$ which we write as $a+ib$.

Then $a+ib = (a,b)$.

Note that if $a+ib \neq 0$, (ie. if $(a,b)\neq(0,0)$)

$$(a+ib)(a-ib) = a^2 - i^2 b = a^2+b^2 \neq 0.$$

So $\Rightarrow$ $\boxed{(a+ib)\cdot \frac{1}{a^2+b^2}(a-ib) = 1}$ $(*)$

$\Rightarrow$ if $a+ib \in \mathbb{C}$ and $a+ib \neq 0$ then $a+ib$ has a multiplicative inverse, ie is a <u>unit</u>.

<u>Def</u> The <u>complex conjugate</u> $\bar{z}$ of $z = a+ib$ is $\bar{z} = a-ib$.

The <u>length</u> (or <u>absolute value</u>) of $z = a+ib$ is

$$|z| = \sqrt{a^2+b^2}$$

<u>Note</u> $|z|^2 = a^2+b^2 = (a+ib)(a-ib) = z\cdot\bar{z}.$

<u>Lemma</u> For any $z, w \in \mathbb{C}$

(1) $\overline{z+w} = \bar{z}+\bar{w}$

(2) $\overline{zw} = \bar{z}\cdot\bar{w}$

<u>Proof</u> (1) exercise. (2) $z = a+ib$, $w = c+id$ for some $a,b,c,d \in \mathbb{C}$. Then

$$\overline{zw} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)} =$$

$$= (ac-bd) - i(ad+bc).$$

On the other hand

$$\bar{z}\cdot\bar{w} = (a-ib)(c-id) = ac + (-i)^2 bd - i(bc+ad)$$

$$= (ac-bd) - i(ad+bc) \qquad \square$$

Note: We can re-write $(*)$ as: if $z \neq 0$ then

$$z^{-1} = \frac{\bar{z}}{z\bar{z}}.$$

Real and imaginary parts: For $z = a+ib$, a is the **real part** of z, and b is the **imaginary part**

$$a = \mathrm{Re}(z), \quad b = \mathrm{Im}(z).$$

Thus $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$.

Note if $z = a+ib$

$$z + \bar{z} = (a+ib) + (a-ib) = 2a$$

$$z - \bar{z} = (a+ib) - (a-ib) = 2ib$$

$$\Rightarrow \quad \mathrm{Re}(z) = \tfrac{1}{2}(z+\bar{z}) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z-\bar{z}) = \frac{-i}{2}(z-\bar{z})$$

Polar form: $(a,b) = \sqrt{a^2+b^2}\,(\cos\theta, \sin\theta)$

where $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$

$$\Rightarrow \quad z = |z|(\cos\theta + i\sin\theta)$$

where $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\mathrm{Im}\,z}{\mathrm{Re}\,z}\right)$

We can either **define** $e^{i\theta} = \cos\theta + i\sin\theta$

or define $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$

and compare it with the power series with $\cos\theta + i\sin\theta$.

Addition formulas for $\sin(\theta_1+\theta_2)$, $\cos(\theta_1+\theta_2)$ translate into

$$(**) \quad e^{i(\theta_1+\theta_2)} = e^{i\theta_1}\cdot e^{i\theta_2}$$

Note $\mathbb{C}^\times = \{z \in \mathbb{C} \mid z \neq 0\}$ is a **group** under multiplication ($1_{\mathbb{C}}$ = the identity in $\mathbb{C}^\times$)

$(**)$ says

$$f : (\mathbb{R}, +, 0) \longrightarrow (\mathbb{C}^\times, \cdot, 1)$$

$$f(\theta) = e^{i\theta} \text{ is a homomorphism.}$$

Note also: $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

Thus $\boxed{\overline{e^{i\theta}} = e^{-i\theta}}$

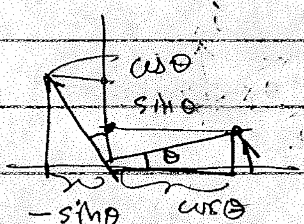
<u>Geometry</u>: <u>Claim</u> The map $f(z) = e^{i\theta} z$ is a rotation θ radians counter-clockwise.

<u>Proof</u> We compute: $e^{i\theta}(a+ib) = (\cos\theta + i\sin\theta)(a+ib) =$

$$= (a\cos\theta - b\sin\theta) + i(a\sin\theta + b\cos\theta)$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{pmatrix}$$

<u>Picture</u>



<u>Ex</u> $f(z) = e^{2\pi i/3} z$ is a $120°$ rotation counterclockwise.

$\tau(z) = \bar{z}$ is a reflection in the x-axis.

<u>The dihedral group D_n</u>, symmetries of the regular n-gon.

Vertices of the n-gon: $\{1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots e^{2k\pi i/n}, \ldots e^{2(n-1)\pi i/n}\}$

D_n is generated by $\rho(z) = e^{2\pi i/n}$ and $\tau(z) = \bar{z}$

<u>Note</u> $\rho^n = \overbrace{\rho \circ \cdots \circ \rho}^{n} = \mathrm{id}$ <u>since</u> $\rho^n z = (e^{2\pi i/n})^n z = z$

$\tau^2 = \tau \circ \tau = \mathrm{id}$ <u>since</u> $\overline{(\bar{z})} = z$.

$(\tau \circ \rho^{-1})(z) = \overline{e^{-2\pi i/n} z} = e^{2\pi i/n}\bar{z} = (\rho \circ \tau)(z)$.

<u>Conclusion</u> $D_n = \langle \rho, \tau \mid \rho^n = \tau^2 = \mathrm{id}, \ \tau\rho^{-1} = \rho\tau \rangle$