10/03/2018
We
14.1

<u>Last time</u> Cosets:

Given a subgroup H of a group G we defined two relations:

$a \sim_\ell b \iff b = ah$ for some $h \in H$
$a \sim_r b \iff b = ha$ for some $h \in H$.
} both are equivalence relations!

The equivalence classes of $\sim_\ell$ are <u>left</u> cosets of H:

$$[a]_\ell := \{ah \mid h \in H\} =: aH$$
$$[a]_r := \{ha \mid h \in H\} = Ha$$

$G/H = \{aH \mid a \in G\}$ the set of left cosets of H

$H\backslash G = \{Ha \mid a \in G\}$ the set of right cosets of H.

We proved Lagrange's Theorem: For a finite group G

$$|G| = |H|\,|G/H|$$

Same argument shows:

$$|G| = |H| \cdot |H\backslash G|.$$

<u>In general</u> the inverse map $inv: G \to G$, $g \mapsto g^{-1}$ gives rise to a bijection

$$\begin{aligned} G/H &\longrightarrow H\backslash G \\ aH &\longmapsto Ha^{-1} \end{aligned}$$

<u>Aside</u> You proved on last homework:

for an equivalence relation $\sim$ on a set X

$$[a] = [b] \iff a \sim b.$$

<u>Solution</u> Suppose $[a] = [b]$. Then, since $a \in [a]$, $a \in [b]$
$\Rightarrow a \sim b$, (since $[b] = \{x \mid x \sim b\}$.)

Conversely, if $a \sim b$, $a \in [b]$. Since $a \in [a]$, $a \in [a] \cap [b]$.
$\Rightarrow [a] \cap [b] \neq \emptyset$. But $[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b]$.
Thus $a \sim b \Rightarrow [a] = [b]$

In general $G/H \neq H\backslash G$

For example: $G = S_3$, $H = \langle (12) \rangle = \{e, (12)\}$

$G/H = \{H, (13)H = \{(13), (123)\}, (23)H = \{(23), (132)\}\}$

$H\backslash G = \{H, H(13) = \{(13), (132)\}, H(23) = \{(23), (123)\}\}$

Why is the map $f: G/H \to H\backslash G$, $f(aH) = Ha^{-1}$ well-defined?

Suppose $aH = bH$. Then $a = bh$ for some $h \in H$

$\Rightarrow a^{-1} = h^{-1}b^{-1} \Rightarrow Ha^{-1} = Hb^{-1}$

Thus $aH = bH \Rightarrow f(aH) = f(bH)$ ✓

□

<u>Proposition 14.1</u> A subgroup N of a group G is normal $\Leftrightarrow$

$Ng = gN$ for all $g \in G$.

<u>Proof</u> ($\Rightarrow$) Suppose N is normal: $\forall g \in G, n \in N$ $gng^{-1} \in N$.

Suppose $x \in Ng$. Then, for some $n \in N$, $x = ng = gg^{-1}ng$.

Since N is normal, $g^{-1}ng \in N$. $\Rightarrow x \in gN$

$\Rightarrow Ng \subseteq gN$. Similarly $gN \subseteq Ng$.

$\therefore Ng = gN$.

($\Leftarrow$) Suppose $gN = Ng$ for all $g \in G$

Then, for any $n \in N$, $gn \in gN = Ng$. $\Rightarrow \exists n' \in N$ s.t.

$gn = n'g$.

$\Rightarrow gng^{-1} = n' \in N$.

$\Rightarrow N$ is normal in G.

<u>Notation</u> $N \triangleleft G$ iff N is a normal subgroup of G.

<u>Corollary 14.2</u> Suppose N is a subgroup of a group G and G/N has exactly 2 elements: $|G/N| = 2$

Then $N \triangleleft G$.

<u>Proof</u> Since $|G/N| = 2$ and $\begin{array}{l} G/N \to N\backslash G \\ gN \mapsto Ng^{-1} \end{array}$
is a bijection, $|N\backslash G| = 2$.
Now since cosets partition G,
$$G/N = \{N,\ G\setminus N = \{g\in G \mid g\notin N\}\}$$
$G \setminus N$ is the coset of $g_0 \notin N$, $G\setminus N = g_0 N$.
Since $g_0 \notin N$, $Ng_0 \neq N$
Since $|N\backslash G| = 2$, $N\backslash G = \{N, Ng_0\}$.
$\Rightarrow$ $Ng_0 = G\setminus N = g_0 N$
By Thm 14.1 $N \lhd G$. $\square$

<u>Ex</u> $K = \langle(123)\rangle$ in S_3 is normal since
$$|S_3/K| = \frac{|S_3|}{|K|} = \frac{6}{3} = 2.$$

<u>Theorem</u> (2.7.1 in Goodman) Suppose N is a normal subgroup of a group G. Then the set G/N of cosets of G has a unique multiplication that makes G/N into a group and
$$\pi : G \to G/N, \quad \pi(g) = gN$$
into a (surjective) homomorphism.

<u>Proof</u> For π to be a homomorphism we must have
$$\pi(ab) = \pi(a)\pi(b) \qquad \forall a, b \in G.$$
ie. $$abN = (aN)\cdot(bN)$$
So if π is a homomorphism, the multiplication $\cdot$ on G/N <u>must</u> be given by
$$(\star) \quad (aN)\cdot(bN) = abN$$
Is $\star$ well-defined? If $aN = a'N$, $bN = b'N$, is it true that $(ab)N = (a'b')N$?
$aN = a'N \Leftrightarrow a' = an_1$ for some $n_1 \in N$
$bN = b'N \Leftrightarrow b' = bn_2$ for some $n_2 \in N$.
$$\Rightarrow \quad a'b' = an_1 bn_2 = abb^{-1}n_1 bn_2 = ab(\underbrace{b^{-1}n_1(b^{-1})^{-1}}_{\in N} n_2)$$

$\Rightarrow a'b' = ab \cdot n$ for $n = (b^{-1} n_1 b) \cdot n_2 \in N$

$\Rightarrow a'b'N = abN.$ (Recall: $x \sim y \Leftrightarrow [x] = [y]$)

$\therefore$ $\cdot : G/N \times G/N \to G/N, \quad (aN, bN) \mapsto (ab)N$

is well-defined.

We set $e_{G/N} = eN (= N)$. Remains to check: $(G/N, \cdot, e_{G/N})$ is a group.

1) $eN \cdot eN = (e \cdot e)N = eN$ ✓

$(aN) \cdot eN = (ae)N = aN, \quad (eN) \cdot (aN) = (ea)N = aN \quad \forall aN \in G/N$

2) $(aN) \cdot (a^{-1}N) = (aa^{-1})N = eN$ and $(a^{-1}N) \cdot (aN) = eN \quad \forall aN \in G/N$

3) $\forall aN, bN, cN \in G/N$

$aN \cdot (bN \cdot cN) = aN \cdot (bc)N = a(bc)N = (ab)cN$

$= ((ab)N) \cdot cN = (aN \cdot bN) \cdot cN.$ □

Def Let N be a normal subgroup of G. The group G/N is called the <u>quotient group</u> (of G by N) and $\pi : G \to G/N$ is the quotient map.

Ex $G = (\mathbb{Z}, +, 0)$ $\quad H = n\mathbb{Z} \quad (n > 1)$

$G/H = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n.$

"Ex" G any group $\quad H = \{e\}.$

H is normal: $\forall g \in G \quad geg^{-1} = e$

What are the cosets of H?

$aH = \{ah \mid h \in H\} = \{a\}$, singletons!

$\pi : G \to G/H$ is $\pi(a) = \{a\}.$

Note π is an <u>isomorphism</u> since it's a homomorphism, it's onto <u>and</u> it's 1-1.

$\Rightarrow G/\{e\}$ is isomorphic to G. We write $G \cong G/\{e\}.$

Special case: $G = \mathbb{Z} \quad H = \{0\}. \quad \mathbb{Z}/\{0\} \cong \mathbb{Z}.$