Last time . $\mathbb{Z}_n$ is an example of a commutative ring

- $9 \mid a \iff 9 \mid$ sum of digits of a $\forall a \in \mathbb{N}$
- Every $[a] \in \mathbb{Z}_n$, $[a] \neq [0]$ is either a unit or a zero divisor

(In $\mathbb{Z}$ there are no zero divisors. The only units are 1 and -1
In $\mathbb{Q}$ there are no zero divisors and any $x \in \mathbb{Q}$, $x \neq 0$ is a unit.)

- $[a] \in \mathbb{Z}_n$ is a unit $\iff \gcd(a, n) = 1$.

Polynomial rings over $K = \mathbb{Q}, \mathbb{R}$ (or $\mathbb{C}$)

Notation

$$K[x] = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid n \geq 0,\ a_0, \dots a_n \in K\}.$$

Polynomials can be added:

$$(a_0 + a_1 x + \dots + a_n x^n) + (b_0 + b_1 x + \dots + b_n x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) x + \dots + (a_n + b_n) x^n$$

Polynomials can be multiplied:

$$(a_0 + \dots + a_n x^n) \cdot (b_0 + b_1 x + \dots + b_m x^m)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$$
$$+ \Big(\sum_{i+j=k} a_i b_j\Big) x^k + \dots + a_n b_m x^{n+m}.$$

$\rightsquigarrow$ $K[x]$ is a commutative ring.

Def 1.8.3 The degree of $p(x) = a_0 + a_1 x^1 + \dots + a_n x^n \in K[x]$ is the largest integer k so that $a_k \neq 0$.

Ex $\deg(1 + 2x + 3x^2 + 0 \cdot x^3) = 2$
$\deg(2) = 0$

Convenient convention $\deg(0) = -\infty$.

<u>Proposition 1.8.5</u> Let $f, g \in K[x]$ be two polynomials.

(a) $\deg(f \cdot g) = \deg f + \deg g$

(b) $\deg(f+g) \leq \max(\deg f, \deg g)$.

<u>Note</u> 1) if $f=0$, $f \cdot g = 0 \quad \forall\, g \in K[x]$

so we want: $\deg(0) = \deg(0) + \deg(g) \quad (*)$

for <u>any</u> $\deg g$, if we set $\deg(0) = -\infty$

and agree that $-\infty + \deg g = -\infty \quad \forall g$

then (a) holds even when $f=0$.

2) $\deg f = 0 \iff f$ is a constant polynomial.

3) Why (b)?

$f(x) = 1+x, \quad g(x) = 2-x$

$f(x)+g(x) = 1+x+2-x = 3 \leq \max\{\deg f, \deg g\}$.

<u>Proof of 1.8.5</u> (a) We may assume $f, g \neq 0$. Then

$f(x) = a_0 + \cdots + a_n x^n$ with $a_n \neq 0$

$g(x) = b_0 + \quad + b_m x^m$ with $b_m \neq 0$

$f(x) \cdot g(x) = a_n b_m \; x^{n+m} +$ lower order terms.

and $a_n b_m \neq 0$ since $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ have no zero divisors.

$\Rightarrow \deg(f \cdot g) = n+m = \deg f + \deg g$.

(b) Let f, g be as before. May assume $n \geq m$.

If $n > m$, $(f+g)(x) = (a_0+b_0) + (a_1+b_1)x^1 + \cdots + (a_m+b_m)x^m + a_{m+1}x^{m+1} + \cdots + \cdots + a_n x^n$

$\Rightarrow \deg(f+g) = n = \max(\deg f, \deg g)$

if $n = m$

$(f+g)(x) = (a_0+b_0) + \cdots + (a_n+b_n)x^n$

$\deg f+g \leq n = \max(\deg f, \deg g)$ since $a_n + b_m$ may be $\underline{\underline{0}}$.

Just as in the case of integers there is a division "algorithm" for polynomials with coefficients in $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

<u>Proposition 9.1</u> (compare with 1.8.13)
Let $p(x), d(x) \in K[x]$ be two polynomials, $\deg d \geq 0$ (so $d(x) \neq 0$). Then there are unique polynomials $q(x), r(x)$ st.

1) $p(x) = d(x)\, q(x) + r(x)$
2) $\deg r(x) < \deg d(x)$

<u>Proof</u> (existence) If $\deg p < \deg d$ then
$$p(x) = 0 \cdot d(x) + p(x) \qquad (\text{so } q = 0,\ r = p).$$
Now suppose $n = \deg p \geq \deg d$.
Induction on n.
If $\underline{n=0}$, $p(x) = b_0$. Since $\deg d \leq \deg p$ and $d \neq 0$, $\deg d = 0$ as well. $\Rightarrow d(x) = a_0$ for some $a_0 \in K$.
$$b_0 = \underbrace{\frac{b_0}{a_0}}_{q} \cdot a_0 + \underbrace{0}_{r}$$

<u>Inductive step</u> $n = \deg p > 0$ and proposition holds for all $h(x) \in K[x]$ with $\deg h < n$.
$$p(x) = a_0 + a_1 x^1 + \quad + a_n x^n, \quad a_0 .. a_n \in K, \quad a_n \neq 0$$
$$d(x) = b_0 + b_1 x^1 + \ - \ + b_m x^m, \quad b_0, .. b_m \in K, \quad b_m \neq 0$$
Consider
$$h(x) = p(x) - \frac{a_n}{b_m} \cdot x^{n-m} d(x) =$$
$$= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) - \frac{a_n}{b_m} x^{n-m} (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0)$$
$$= \underbrace{\left(a_n - \frac{a_n}{b_m} b_m\right)}_{=0} x^n + \left(a_{n-1} - \frac{a_n}{b_m} b_{m-1}\right) x^{n-1} + \cdots + a_0$$
$\Rightarrow \deg h \leq n-1.$

By the inductive assumption there exist polynomials $q_1(x)$, $r_1(x)$ so that $h(x) = q_1(x) \cdot d(x) + r_1(x)$
and $\deg r_1(x) < \deg d(x)$.

$$\Rightarrow \quad p(x) - \frac{a_n}{b_m} x^{n-m} \cdot d(x) = q_1(x)\, d(x) + r_1(x)$$

$$\Rightarrow \quad p(x) = \left(\frac{a_n}{b_m} x^{n-m} + q_1(x)\right) d(x) + r_1(x)$$

Uniqueness (not in Goodman)

Suppose $p(x) = q_1(x)\, d(x) + r_1(x) = q_2(x)\, d(x) + r_2(x)$,
and $\deg r_1, \deg r_2 < \deg d$.

Then

$$(q_1 - q_2) \cdot d = r_2 - r_1$$

$$\Rightarrow \quad \deg(q_1 - q_2) + \deg d = \deg(r_2 - r_1) \le \max(\deg r_1, \deg r_2) < \deg d$$

$$\Rightarrow \deg(q_1 - q_2) < 0$$

$$\Rightarrow \deg(q_1 - q_2) = -\infty, \text{ ie } q_1 - q_2 = 0.$$

$$\Rightarrow r_1 - r_2 = 0 \text{ as well}$$ $\square$

Ex

$$\begin{array}{r|l} & 3x^2 + 6x + 5 \\ \hline x - 1 & 3x^3 + 3x^2 - x + 1 \\ & \underline{3x^3 - 3x^2} \\ & \quad 6x^2 - x \\ & \quad \underline{6x^2 - 6x} \\ & \qquad 5x + 1 \\ & \qquad \underline{5x - 5} \\ & \qquad\quad 6 \end{array}$$

$$\Rightarrow \quad 3x^3 + 3x^2 - x + 1 = (x-1)(3x^2 + 6x + 5) + 6.$$